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Penalty-Free Discontinuous Galerkin Methods for the Stokes and Navier-Stokes Equations

by

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ABSTRACT

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This thesis formulates and analyzes low-order penalty-free discontinuous Galerkin methods for solving the incompressible Stokes and Navier-Stokes equations. Some symmetric and non-symmetric discontinuous Galerkin methods for incompressible Stokes and Navier-Stokes equations require penalizing jump terms for stability and convergence of the methods. These discontinuous Galerkin methods are called interior penalty methods as the penalizing jump terms involve a penalty parameter. It is known that the penalty parameter has to be large enough to prove coercivity of the bilinear form and therefore to obtain existence of the solution for the symmetric case. The momentum equation is satisfied locally on each mesh element, and it depends on the penalty parameter. Setting the penalty parameter equal to zero yields a singular linear system, if piecewise linears are used. To overcome this instability, this thesis discusses an enrichment of the velocity space with locally supported quadratic functions called bubbles.

First, the penalty-free non-symmetric discontinuous Galerkin method is analyzed for the Stokes equations. Second, the main contribution of this thesis is the analysis of both symmetric and non-symmetric penalty-free discontinuous Galerkin methods for the incompressible Navier-Stokes equations. Since a direct application of the generalized Lax-Milgram theorem is not possible, the numerical solution is shown to be the solution as a fixed-point of a problem-related map. A priori error estimate is derived.

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Chapter 1

Introduction

1.1 Introduction

The Stokes and incompressible Navier-Stokes equations are used to model flow of fluid with low Mach and Reynolds numbers. The Stokes equations are used when the advective inertia forces are small compared to viscous forces and therefore, they can be neglected. In the case that advective inertia forces cannot be neglected we use the Navier-Stokes equations.

This thesis formulates and analyzes low-order penalty-free discontinuous Galerkin methods for solving the incompressible steady Stokes and Navier-Stokes equations. Some symmetric and non-symmetric discontinuous Galerkin methods for incompressible Stokes and Navier-Stokes equations require penalizing jump terms for stability and convergence of the methods. These discontinuous Galerkin methods are called interior penalty methods as the penalizing jump terms involve a penalty parameter. It is known that the penalty parameter has to be large enough to prove the coercivity of the symmetric case and the issue of finding optimal penalty parameter needs more investigation. However the symmetric formulation yields a symmetric scheme which is adjoint consistent. On the other hand, in the non-symmetric case, the penalty parameter can be set equal to one but the method is not adjoint consistent. In both methods the momentum equation is satisfied locally on each mesh element, and it depends on the penalty parameter. Setting the penalty parameter to zero yields a singular linear system, if piecewise linears are used. To overcome this instability, this thesis discusses an enrichment of the velocity space with locally supported quadratic functions called bubbles.

This thesis is organized as follows. In chapter 2, I present the model problem for the Stokes equations and define the spaces in which the problem is stated. In this chapter I give the definition of the mesh, jumps and averages for scalar or vector fields. This chapter introduces the velocity and pressure space and the norms associated to these spaces. Also, the non-symmetric discrete problem, interpolations, projections and technical lemmas that are used to prove the main results are given. The analysis of the existence and uniqueness of the solution for Stokes problem and error estimation is derived.

The main results of this thesis are given in chapter 3. Chapter 3 states the model problem for the steady incompressible Navier-Stokes equations. In this chapter existence of the numerical solution for symmetric and non-symmetric penalty-free DG scheme is proved. At the end of this chapter we provide error estimates for both symmetric and non-symmetric cases. I then present some numerical results that confirm the theoretical error estimates.

1.2 Literature Review

Methods for solving the incompressible steady state Navier-Stokes equations include [32,28,18,19], [33] finite volume methods [20,46,23], finite element methods [24,27,38] and discontinuous Galerkin methods [15,13,14,12,31,35,36,34,26,40].

Finite difference methods are easy to implement but are not appropriate for complex geometries. Finite volume methods are conservative methods but are usually first order accurate and ineffective for discontinuous and tensor coefficient problems. Finite element methods are widely used for computational fluid dynamics because they are accurate, much more stable than finite volume methods and can handle the complex geometries very well. One disadvantage of the finite element is that they are not inherently locally conservative methods. To obtain an independent and consistent system of equations for Stokes and Navier-Stokes, the number of pressure unknowns should never exceed the number of velocity unknowns independently of the number of elements [43]. Therefore, admissible elements or splitting schemes should be chosen. Some examples of admissible elements are the Taylor-Hood family [44], the Crouzeix-Raviart family [16, 17] and the MINI elements [3].

Since discontinuous Galerkin methods use discontinuous approximations, they easily can handle discontinuous coefficients, irregular meshes with hanging nodes and arbitrary shapes. With the proper choice of their numerical traces they become locally conservative. Therefore, discontinuous Galerkin (DG) methods enjoy properties of both finite volume methods and finite element methods. Also, for Stokes and Navier-Stokes equations the polynomial basis of any order can be chosen without worrying about admissibility of the elements. These properties make the discontinuous Galerkin methods a good candidate for computational fluid dynamics.

In the literature there are several DG methods for solving incompressible Stokes and Navier-Stokes equations. They include local discontinuous Galerkin methods (LDG) [15,13,14], compact discontinuous Galerkin methods (CDG) [34], hybridizable discontinuous Galerkin methods (HDG) [12, 31, 35, 36], and interior penalty discontinuous Galerkin methods (SIPG-DG, NIPG-DG, OBB-DG) [26, 40].

Local discontinuous Galerkin methods are stable, locally conservative, high order methods. Cockburn, Kanschat, Schotzau and Schwab analyzed the LDG method for the Stokes system in 2002 [15]. In 2003 the LDG method for Oseen problem was introduced [13] and in 2004 Cockburn et al proposed the LDG method for the incompressible Navier-Stokes equations [14] that is divergence free.

Hybridizable Discontinuous Galerkin (HDG) methods produce a final system in terms of the degrees of freedom of the approximate traces of the field variables rather than a final system involving the degrees of freedom of the approximate field variables. Since the approximate traces depend on the element faces only and are single-valued on every face, the HDG methods have significantly less globally coupled unknowns than other DG methods. This can lead to significant savings for both computational time and memory storage. Some works on HDG for Stokes problem can be found in [12,31]. In [35,36] several HDG methods for incompressible Navier-Stokes equations are proposed.

The compact discontinuous Galerkin methods are introduced by Peraire and Persson in [29] for elliptic problems. In [34] Montlaue formulates the compact discontinuous Galerkin analysis for incompressible Navier-Stokes equations. One advantage of the CDG method is a sparser matrix.

Another class of DG methods for incompressible Stokes and Navier-Stokes equations are the interior penalty discontinuous Galerkin methods. Symmetric interior penalty (SIPG), non-symmetric interior penalty (NIPG) and Oden-Baumann-Babuska (OBB) are three methods in this class. These methods have been applied to the incompressible Stokes and Navier-Stokes problems (see for instance [26] and [40]). However the theoretical analysis was done for SIPG and NIPG only. The analysis requires the use of penalty terms.

Interior penalty DG methods were first introduced for elliptic problems [4]. At the same time, the interior penalty method of Douglas and Dupont for second order elliptic problems [21] led to the symmetric interior penalty DG method SIPG proposed by Wheeler [45] and Arnold [2]. A slight variation of the SIPG methods yields a non-symmetric formulation, called non-symmetric interior penalty Galerkin methods (NIPG) (see [41]).

The Oden-Baumann-Babuska (OBB) method was introduced for elliptic problems by Oden et al [37]. Analysis of OBB in 2D and 3D was done by Riviere, Wheeler and Girault [41] and by Larson and Niklasson [30]. Because the OBB does not have a penalty term, a special interpolant needs to be constructed to prove convergence of the method.

This thesis proposes a penalty-free method for incompressible Stokes and Navier-Stokes equations so that local mass conservation is independent of penalty parameters. Several attempts have been made recently in designing penalty-free DG methods. First for elliptic problems in [6] Brezzi and Marini showed that the smallest space for which optimal convergence for the non-symmetric version is obtained is the space of piecewise affine functions enriched with quadratic bubbles. Antonietti, Brezzi and Marini [1] extended their work to general grids with n-edge polygonals and showed how to construct the bubbles for quadrilateral grids. Another related work on penalty-free methods is the work of Burman and Stamm [7, 8]. They showed convergence of the stabilized DG method for both OBB method and symmetric version, applied to elliptic and parabolic equations. Burman and Stamm in [9], extended their work to the incompressible Stokes equations and analyzed the symmetric penalty-free DG method.

In this thesis, we consider the same stabilization technique as in [9] and apply it to the incompressible steady Navier-Stokes equations. I also analyze the non-symmetric penalty-free DG method for the Stokes equations.

Chapter 2

Stokes Equations

In this chapter I consider the Stokes equations and formulate the bubble stabilized non-symmetric discontinuous Galerkin methods. The analysis is the modification of the work in [9], where only the symmetric stabilized DG methods was studied. The outline of the chapter is as follows:

I first describe the model problem for the incompressible steady Stokes problem. In section 2.1 I restate the definition of the Sobolev spaces, jumps and averages of functions for interior and boundary functions. The definition of the discrete velocity and pressure spaces and penalty-parameter free norms for these spaces are given in section 2.2. The work follows by section 2.3 which has three subsections 2.3.1, 2.3.2 and subsection 2.3.3. In section 2.3.1 I define the numerical scheme for the non-symmetric penalty free Stokes equations. In sections 2.3.2 and 2.3.3 I restate the technical lemmas found in [9]. Since, only the proof of the theorem 2.3.13 is given in [9], I prove the lemmas in the appendix A. Section 2.4 is the main section of this chapter and has two subsections. In the first part I state the generalized Lax-Milgram theorem and then apply it to my problem to prove the existence and uniqueness of the solution of the non-symmetric penalty-free Stokes. In the second part a priori error estimates are derived. Proofs in this subsection are slight modifications of the proofs in [9]. For completeness I recalled them in this work.

I now describe the model problem. Let Ω be a Lipschitz, and convex polygon in $\mathbb{R}^d$, $d = 2, 3$. The incompressible Stokes equations are as follows: given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$

find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$, such that

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega,$$
$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega,$$
$$\boldsymbol{u} = 0, \quad \text{on } \partial\Omega. \tag{2.1}$$

The Sobolev spaces above are defined as;

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \quad v = 0 \text{ on } \ \partial\Omega\}, \quad \boldsymbol{H}_0^1 = (H_0^1)^d, \tag{2.2}$$
$$L_0^2(\Omega) = \{v \in L^2(\Omega) : \quad \int_\Omega v = 0\}. \tag{2.3}$$

2.1 Notation

In this section I define the Sobolev spaces, jumps and averages of functions for interior and boundary faces. Let $\mathcal{T}_h$ be a shape-regular subdivision of Ω. Let $\mathcal{F}_i$ be the set of interior faces that are not included in $\partial\Omega$ and let $\mathcal{F}_e$ be the set of boundary faces. Also denote $\mathscr{F}_h = \mathcal{F}_i \cup \mathcal{F}_e$ the set of all faces. For an element $\tau \in \mathcal{T}_h$, h_τ denotes its diameter and for a face $F \in \mathscr{F}_h$, h_F denotes the diameter of F. Set $h = \max_{\tau \in \mathcal{T}_h} h_\tau$ and let $\tilde{h}$ be the function such that $\tilde{h}|_\tau = h_\tau$ and $\tilde{h}|_F = h_F$ for all $\tau \in \mathcal{T}_h$ and $F \in \mathscr{F}_h$. The L^2 inner-product on a domain $\mathcal{Q}$ is : $(v, w)_\mathcal{Q} = \int_\mathcal{Q} vw$.
Denote for $s \geq 1$, $H^s(\mathcal{T}_h)$ the space of piecewise Sobolev H^s-functions and denote by $\boldsymbol{H}^s(\mathcal{T}_h)$ the space $(H^s(\mathcal{T}_h))^d$ where

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega) \ : \ \forall \tau \in \mathcal{T}_h \quad v|_\tau \in H^s(\tau)\}.$$

Let $\boldsymbol{v} = (v_1, ..., v_d)^T \in \boldsymbol{H}^s(\mathcal{T}_h)$ then $\nabla \boldsymbol{v}|_\tau \in L^2(\tau)^{d \times d}$ and I define for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1(\mathcal{T}_h)$:

$$(\boldsymbol{v}, \boldsymbol{w})_\tau = \sum_{i=1}^{d} (v_i, w_i)_\tau,$$

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{w})_\tau = \sum_{i,j=1}^{d} (\partial_{x_j} v_i, \partial_{x_j} w_i)_\tau.$$

In addition let us define the jump and average operators. Let F be a fixed face shared by two elements τ_1, τ_2. Assume $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h)$ and $\boldsymbol{v}|_{\tau_1}, \boldsymbol{v}|_{\tau_2}$ are the restriction of $\boldsymbol{v}$ to elements τ_1, τ_2 respectively. Also, suppose $\boldsymbol{n}_1, \boldsymbol{n}_2$ are exterior normals of τ_1, τ_2 respectively. Then I define the average $\{\cdot\}, \{\{\cdot\}\}$ and jump $[\cdot], [[\cdot]]$ operators by:
For interior faces:

$$\{v\} = \frac{1}{2}(v_1 + v_2), \quad [v] = v_1 \boldsymbol{n}_1 + v_2 \boldsymbol{n}_2, \quad \forall v \in H^1(\mathcal{T}_h), \tag{2.4}$$

$$\{\boldsymbol{v}\} = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2), \quad [\boldsymbol{v}] = \boldsymbol{v}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{n}_2, \quad \forall \boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h), \tag{2.5}$$

and

$$\{\{\boldsymbol{v}\}\} = \frac{1}{2}(\boldsymbol{v}_1 + \boldsymbol{v}_2), \quad [[\boldsymbol{v}]] = \boldsymbol{v}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{n}_2, \tag{2.6}$$

$$\{\nabla \boldsymbol{v}\} = \frac{1}{2}(\nabla \boldsymbol{v}_1 + \nabla \boldsymbol{v}_2), \quad [[\nabla \boldsymbol{v}]] = \nabla \boldsymbol{v}_1 \boldsymbol{n}_1 + \nabla \boldsymbol{v}_2 \boldsymbol{n}_2. \tag{2.7}$$

For boundary faces:

$$\{v\} = v, \ \ \{\boldsymbol{v}\} = \boldsymbol{v}, \ \ \{\{\boldsymbol{v}\}\} = \boldsymbol{v}, \ \ \{\{\nabla \boldsymbol{v}\}\} = \nabla \boldsymbol{v}, \tag{2.8}$$

$$[v] = v\boldsymbol{n}, \ \ [\boldsymbol{v}] = \boldsymbol{v} \cdot \boldsymbol{n}, \ \ [[\boldsymbol{v}]] = \boldsymbol{v} \otimes \boldsymbol{n}, \ \ [[\nabla \boldsymbol{v}]] = \nabla \boldsymbol{v} \boldsymbol{n}. \tag{2.9}$$

I recall that for two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$, the product $\boldsymbol{a} \otimes \boldsymbol{b}$ is a matrix in $\mathbb{R}^{d \times d}$ with entries $a_i b_j$. The vector $\boldsymbol{n}$ is the normal to the domain.

I also recall that for two matrix functions $\boldsymbol{A}, \boldsymbol{B} : \mathbb{R}^d \to \mathbb{R}^{d\times d}$, I have

$$\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^{d} a_{ij} b_{ij}.$$

I also define

$$(\boldsymbol{A}, \boldsymbol{B})_\tau = \int_\tau \boldsymbol{A} : \boldsymbol{B}.$$

In this thesis, a generic constant independent of h is defined by M. The constant M may take different values at different places.

2.2 Discrete Spaces and Norms

In this section, I define the discrete velocity and pressure spaces and I equip these spaces with penalty-parameter free norms.

2.2.1 Velocity Space

Let V_h^p denote the space of piecewise polynomials of degree $p \geq 0$ i.e.

$$V_h^p = \{v_h \in L^2(\Omega) : v_h|_\tau \in \mathbb{P}_p(\tau), \;\; \forall \tau \in \mathcal{T}_h\}$$

where $\mathbb{P}_p(\tau)$ denotes the set of polynomials of maximum degree p on τ. I also define the space enriched by quadratic functions called bubbles and denoted by V_{bs}

$$V_{bs} = V_h^1 \oplus \{v_h \in L^2(\Omega) : v_h(\boldsymbol{x}) = \alpha \boldsymbol{x}.\boldsymbol{x}, \alpha \in V_h^0\}.$$

Moreover, let us define a discrete space on the set of faces:

$$W_h^0 = \{v_h \in L^2(\mathscr{F}_h) : v_h|_F \in \mathbb{P}_0(F), \forall F \in \mathscr{F}_h\}.$$

The corresponding vectorial versions are

$$\boldsymbol{V}_h^p = (V_h^p)^d \;\;, \;\; \boldsymbol{V}_{bs} = (V_{bs})^d, \;\; \text{and } \boldsymbol{W}_h^0 = (W_h^0)^d.$$

The spaces $\boldsymbol{V}_h^1$, $\boldsymbol{V}_{bs}$ are equipped with the norm:

$$|||\, \boldsymbol{v} \,|||^2 = ||\, \nabla \boldsymbol{v} \,||^2_{\mathcal{T}_h} + ||\, \widetilde{h}^{-\frac{1}{2}}[[\boldsymbol{v}]] \,||^2_{\mathscr{F}_h}, \tag{2.10}$$

where

$$\begin{aligned} ||\nabla \boldsymbol{v}||^2_{\mathcal{T}_h} &= \sum_{\tau \in \mathcal{T}_h} ||\nabla \boldsymbol{v}||^2_{\tau}, \\ ||\tilde{h}^{-\frac{1}{2}}[[\boldsymbol{v}]]||^2_{\mathscr{F}_h} &= \sum_{F \in \mathscr{F}_h} ||\, \frac{1}{\tilde{h}^{\frac{1}{2}}}[[\boldsymbol{v}]] \,||^2_F \,. \end{aligned}$$

Remark 2.2.1. *The norm $|||\cdot|||$ is the usual DG norm, as I note that $||[[\boldsymbol{v}_h]]||_F = ||j(\boldsymbol{v}_h)||_F$ for all faces F where $j(\boldsymbol{v}_h)$ is the usual jump:*

$$\begin{aligned} j(\boldsymbol{v}) &= \boldsymbol{v}|_{\tau_1} - \boldsymbol{v}|_{\tau_2} \quad \textit{on interior faces} \\ j(\boldsymbol{v}) &= \boldsymbol{v}|_{\tau_1} \quad \textit{on boundary faces.} \end{aligned}$$

2.2.2 Pressure Space

I can choose piecewise constants, piecewise linears or a combination of the two for the pressure space. Mathematically I denote the pressure space by $Q_h = \alpha Q^1_{h,c} \oplus \beta Q^0_{h,d}$ where $\alpha, \beta \in \{0, 1\}$ and,

$$Q^1_{h,c} = \{v_h \in C^0(\bar{\Omega}) \cap L^2_0(\Omega) : v_h|_\tau \in \mathbb{P}_1(\tau), \forall \tau \in \mathcal{T}_h\}, \tag{2.11}$$

$$Q^0_{h,d} = \{v_h \in L^2_0(\Omega) : v_h|_\tau \in \mathbb{P}_0(\tau), \forall \tau \in \mathcal{T}_h\}. \tag{2.12}$$

The space Q_h is equipped with the norm:

$$\forall q_h \in Q_h \quad |||\, q_h \,|||_Q^2 = ||\widetilde{h}\nabla q_h \,||_{\mathcal{T}_h}^2 + ||\widetilde{h}^{\frac{1}{2}}[q_h]\,||_{\mathcal{F}_i}^2. \tag{2.13}$$

I also equip the space $\boldsymbol{V}_{bs} \times Q_h$ with the triple norm

$$|||(\boldsymbol{v}_h, q_h)|||^2 = |||\boldsymbol{v}_h|||^2 + |||q_h|||_Q^2. \tag{2.14}$$

2.3 Discretized Stokes Equations by Non-Symmetric DG Method

This section introduces the non-symmetric scheme for the Stokes equation. Several technical lemmas are stated, the proof of which is found in appendix A.

2.3.1 Numerical Solution

The diffusive term of the Stokes equations is discretized by the following bilinear form

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_{\mathcal{T}_h} - (\{\nabla \boldsymbol{u}_h\}, [[\boldsymbol{v}_h]])_{\mathscr{F}_h} + ([[\boldsymbol{u}_h]], \{\nabla \boldsymbol{v}_h\})_{\mathscr{F}_h}, \tag{2.15}$$

or equivalently,

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_{\mathcal{T}_h} - (\{\nabla \boldsymbol{u}_h\}\boldsymbol{n}_\tau, j(\boldsymbol{v}_h))_{\mathscr{F}_h} + (j(\boldsymbol{u}_h), \{\nabla \boldsymbol{v}_h\}\boldsymbol{n}_\tau)_{\mathscr{F}_h},$$

which is the standard form given in [26]. The pressure term discretization is defined by the following form

$$b(p_h, \boldsymbol{v}_h) = -(p_h, \nabla \cdot \boldsymbol{v}_h)_{\mathcal{T}_h} + (\{p_h\}, [\boldsymbol{v}_h])_{\mathscr{F}_h}. \tag{2.16}$$

Finally the right-hand side is given by

$$F(\boldsymbol{v}_h, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h)_{\mathcal{T}_h}. \tag{2.17}$$

Numerical scheme: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) = F(\boldsymbol{v}_h, q_h) \tag{2.18}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$.

2.3.2 Technical Lemmas

In this part I state a few technical lemmas related to jump and average that I later use to build the main interpolation operator. Let $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h)$ and let F be a fixed face that belongs to $\partial\tau_1$ and $\partial\tau_2$ for some elements τ_1, τ_2 in $\mathcal{T}_h$.

Lemma 2.3.1. *If $\boldsymbol{n}_F$ is chosen arbitrary ($\boldsymbol{n}_F \in \{\boldsymbol{n}_1, \boldsymbol{n}_2\}$), then I have*

$$[[\boldsymbol{v}]] : \nabla \boldsymbol{w} = [[\boldsymbol{v}]]\boldsymbol{n}_F \cdot \nabla \boldsymbol{w} \boldsymbol{n}_F, \tag{2.19}$$

$$[[\boldsymbol{v}]] = [[\boldsymbol{v}]]\boldsymbol{n}_F \otimes \boldsymbol{n}_F, \quad [\boldsymbol{v}] = [[\boldsymbol{v}]]\boldsymbol{n}_F \cdot \boldsymbol{n}_F. \tag{2.20}$$

Lemma 2.3.2.

$$\| \, [[\boldsymbol{v}]] \, \|_{\mathscr{F}_h} = \| \, [[\boldsymbol{v}]]\boldsymbol{n}_F \, \|_{\mathscr{F}_h}, \quad \| \, [\boldsymbol{v}] \, \|_{\mathscr{F}_h} \leq \sqrt{3} \, \| \, [[\boldsymbol{v}]] \, \|_{\mathscr{F}_h} . \tag{2.21}$$

Lemma 2.3.3. *For $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ I have that $\Delta \boldsymbol{v}_h \in \boldsymbol{V}_h^0$ and $\Delta : \frac{\boldsymbol{V}_{bs}}{\boldsymbol{V}_h^1} \to \boldsymbol{V}_h^0$ is bijective.*

The Raviart-Thomas space of order 0 is defined as usual by $RT_0(\mathcal{T}_h)$ and its vectorial version by $\boldsymbol{RT}_0(\mathcal{T}_h)$.

Lemma 2.3.4. *For all $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ with $\boldsymbol{v}_h = (v_{h,1}, ..., v_{h,d})^T$ there holds*

$$\nabla v_{h,i}|_\tau \in \boldsymbol{RT}_0(\tau), \forall \tau \in \mathcal{T}_h.$$

Besides, for all $\tau \in \mathcal{T}_h$ and $\boldsymbol{r}_h = (\boldsymbol{r}_{h,1}, ..., \boldsymbol{r}_{h,d})$ with $\boldsymbol{r}_{h,i} \in \boldsymbol{RT}_0(\tau)$, there exists $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ such that $\nabla v_{h,i}(\tau) = \boldsymbol{r}_{h,i}$ for all $1 \leq i \leq d$.

Corollary 2.3.5. *If $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ then $\{\nabla \boldsymbol{v}_h\}\boldsymbol{n}_F \in \boldsymbol{W}_h^0$ and $[[\nabla \boldsymbol{v}_h]] \in \boldsymbol{W}_h^0$.*

Lemma 2.3.6. *If $v \in H^1(\mathcal{T}_h), \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{H}^1(\mathcal{T}_h)$, then*

$$(v, \nabla \cdot \boldsymbol{w})_{\mathcal{T}_h} = -(\nabla v, \boldsymbol{w})_{\mathcal{T}_h} + (\{v\}, [\boldsymbol{w}])_{\mathscr{F}_h} + ([v], \{\boldsymbol{w}\})_{\mathcal{F}_i}, \tag{2.22}$$

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{w})_{\mathcal{T}_h} = -(\Delta \boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_h} + (\{\nabla \boldsymbol{v}\}, [[\boldsymbol{w}]])_{\mathscr{F}_h} + ([[\nabla \boldsymbol{v}]], \{\boldsymbol{w}\})_{\mathcal{F}_i}, \tag{2.23}$$

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{w})_{\mathcal{T}_h} = -(\Delta \boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_h} + (\{\nabla \boldsymbol{v}\}\boldsymbol{n}_F, [[\boldsymbol{w}]]\boldsymbol{n}_F)_{\mathscr{F}_h} + ([\nabla \boldsymbol{v}], \{\boldsymbol{w}\})_{\mathcal{F}_i}. \tag{2.24}$$

Lemma 2.3.7. Inverse inequality: *Let $m \in \{1, d\}$ $\boldsymbol{v}_h \in (\boldsymbol{V}_{bs})^m$. Then, there exists $C_I > 0$ independent of h such that*

$$C_I^{-1} \parallel \tilde{h}^2 \Delta \boldsymbol{v}_h \parallel_{\mathcal{T}_h} \leq \parallel \tilde{h} \nabla \boldsymbol{v}_h \parallel_{\mathcal{T}_h} \leq \ C_I \parallel \boldsymbol{v}_h \parallel_{\mathcal{T}_h} . \tag{2.25}$$

Theorem 2.3.8. Trace inequalities: *For $m \in \{1, d, d^2\}$, $\boldsymbol{v} \in (\boldsymbol{H}^1_{\mathcal{T}_h})^m$, $\boldsymbol{v}_h \in (\boldsymbol{V}_{bs})^m$, $\boldsymbol{w} \in (\boldsymbol{H}^1(\mathcal{T}_h))^d$, and $\boldsymbol{w}_h \in (\boldsymbol{V}_{bs})^d$, there exists a constant $C_T > 0$, independent of h*

such that

$$\| \{\boldsymbol{v}\} \|_{\mathscr{F}_h} + \| [\boldsymbol{v}] \|_{\mathscr{F}_h} \leq C_T(\| h^{-\frac{1}{2}}\boldsymbol{v} \|_{\mathcal{T}_h} + \| h^{-\frac{1}{2}}\nabla\boldsymbol{v} \|_{\mathcal{T}_h}), \tag{2.26}$$

$$\| \{\boldsymbol{v}_h\} \|_{\mathscr{F}_h} + \| [\boldsymbol{v}_h] \|_{\mathscr{F}_h} \leq C_T \| h^{-\frac{1}{2}}\boldsymbol{v}_h \|_{\mathcal{T}_h}, \tag{2.27}$$

$$\| \{\{\boldsymbol{w}\}\} \|_{\mathscr{F}_h} + \| [[\boldsymbol{w}]] \|_{\mathscr{F}_h} \leq C_T(\| h^{-\frac{1}{2}}\boldsymbol{w} \|_{\mathcal{T}_h} + \| h^{-\frac{1}{2}}\nabla\boldsymbol{w} \|_{\mathcal{T}_h}, \tag{2.28}$$

$$\| \{\{\boldsymbol{w}_h\}\} \|_{\mathscr{F}_h} + \| [[\boldsymbol{w}_h]] \|_{\mathscr{F}_h} \leq C_T \| h^{-\frac{1}{2}}\boldsymbol{w}_h \|_{\mathcal{T}_h} . \tag{2.29}$$

2.3.3 Projections and Interpolants

In this section the Clement interpolant, the Crouzeix-Raviart interpolant and the L^2 projection are introduced. These interpolants and projection are used to prove the convergence results. Finally, an important projection is defined.

1. L^2 projection onto $(\boldsymbol{W}_h^0)^m$ for $m \in \{1, d, d^2\}$ and $\boldsymbol{v} \in [L^2(\mathscr{F}_h)]^m$ is given by $\bar{\boldsymbol{v}}$ and is defined as:

$$(\bar{\boldsymbol{v}}, \boldsymbol{w}_h)_{\mathscr{F}_h} = (\boldsymbol{v}, \boldsymbol{w}_h)_{\mathscr{F}_h}, \quad \forall \boldsymbol{w}_h \in (\boldsymbol{W}_h^0)^m. \tag{2.30}$$

   By choosing $\boldsymbol{w}_h = \bar{\boldsymbol{v}}$ and using Cauchy-Schwarz's inequality, it is easy to show that

$$||\bar{\boldsymbol{v}}||_{\mathscr{F}_h} \leq ||\boldsymbol{v}||_{\mathscr{F}_h}. \tag{2.31}$$

2. L^2 projection onto V_h^p, $\pi_p : L^2(\Omega) \longrightarrow V_h^p$ and its vectorial version (element-wise) $\boldsymbol{\pi}_p$,

$$\int_\Omega (\pi_p v)\, w_h dx = \int_\Omega v\, w_h dx \quad \forall w_h \in V_h^p. \tag{2.32}$$

If $v \in H^{p+1}(\mathcal{T}_h)$ and $\boldsymbol{v} \in \boldsymbol{H}^{p+1}(\mathcal{T}_h)$, then

$$\| v - \pi_p v \|_{\mathcal{T}_h} + \| \widetilde{h}\nabla(v - \pi_p v) \|_{\mathcal{T}_h} \le M h^{p+1} |v|_{p+1,\mathcal{T}_h}, \tag{2.33}$$

$$\| \boldsymbol{v} - \boldsymbol{\pi}_p \boldsymbol{v} \|_{\mathcal{T}_h} + \| \widetilde{h}\nabla(\boldsymbol{v} - \boldsymbol{\pi}_p \boldsymbol{v}) \|_{\mathcal{T}_h} \le M h^{p+1} |\boldsymbol{v}|_{p+1,\mathcal{T}_h}. \tag{2.34}$$

3. The Crouzeix-Raviart interpolant: Let CR and its vectorial version $\boldsymbol{CR}$ be the Crouzeix-Raviart space, and let assume $\boldsymbol{i}_c : \boldsymbol{H}^1(\Omega) \to \boldsymbol{CR}$ be the vectorial Crouzeix-Raviart interpolant then for all $\boldsymbol{v} \in \boldsymbol{H}^2(\Omega)$

$$\| \boldsymbol{v} - \boldsymbol{i}_c \boldsymbol{v} \|_{\mathcal{T}_h} + \| \widetilde{h}\nabla(\boldsymbol{v} - \boldsymbol{i}_c \boldsymbol{v}) \|_{\mathcal{T}_h} \le M h^2 |\boldsymbol{v}|_{2,\mathcal{T}_h} \tag{2.35}$$

for a constant M independent of h.

4. Clement interpolant (definition at [10]): Denoted by $C_h : L^2(\Omega) \to V^1_{h,c}$ where

$$V^1_{h,c} = \{v_h \in C^0(\bar{\Omega}) | v_h|_\tau \in \mathbb{P}_1(\tau) \ \forall \tau \in \mathcal{T}_h\}$$

Then,

$$\| v - C_h v \|_{\mathcal{T}_h} + \| \widetilde{h}\nabla(v - C_h v) \|_{\mathcal{T}_h} \le M h^{\gamma+1} |v|_{\gamma+1,\mathcal{T}_h} \tag{2.36}$$

where $v \in H^{\gamma+1}(\mathcal{T}_h)$, $\gamma \in \{0, 1\}$.

Note that the vectorial Clement interpolant $\boldsymbol{C}_h$ also satisfies (2.36).

5. The projection $P_\alpha : L^2(\Omega) \to Q_h$ is defined by $P_\alpha = (1-\alpha)\boldsymbol{\pi}_0 + \alpha C_h$ and for all $v \in H^{\gamma+1}(\mathcal{T}_h), \gamma \in \{0, 1\}$ has the property,

$$\| v - P_\alpha v \|_{\mathcal{T}_h} + \| \widetilde{h}\nabla(v - P_\alpha v) \|_{\mathcal{T}_h} \le M h^{\alpha\,\gamma+1} |v|_{\alpha\,\gamma+1,\mathcal{T}_h}. \tag{2.37}$$

Lemma 2.3.9.

$$|||[[\bar{\boldsymbol{v}}]]|||_{\mathscr{F}_h} = |||[[\bar{\boldsymbol{v}}]]\boldsymbol{n}_F||_{\mathscr{F}_h}. \tag{2.38}$$

Lemma 2.3.10. *For all* $\boldsymbol{v}_h \in \boldsymbol{H}^1(\mathcal{T}_h)$

$$\| \tilde{h}^{-\frac{1}{2}}[[\boldsymbol{v}_h]] \|^2_{\mathscr{F}_h} \leq M \ (\| \tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{v}}_h]] \|^2_{\mathscr{F}_h} + \| \nabla \boldsymbol{v}_h \|^2_{\mathcal{T}_h}) \tag{2.39}$$

where $M > 0$ *is a constant independent of* h.

Theorem 2.3.11. Poincare inequality: *For all* $\boldsymbol{v}_h \in \boldsymbol{H}^1(\mathcal{T}_h)$ *there holds*

$$||\boldsymbol{v}_h||_{\mathcal{T}_h} \leq C_P|||\boldsymbol{v}_h||| \tag{2.40}$$

where C_P *is a constant independent of* h. *And,*

$$\| \boldsymbol{v}_h \|^2_{\mathcal{T}_h} \leq \tilde{C}_p(\| \tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{v}}_h]] \|^2_{\mathscr{F}_h} + \| \nabla \boldsymbol{v}_h \|^2_{\mathcal{T}_h}) \tag{2.41}$$

where $\tilde{C}_p$ *is a constant independent of* h.

Lemma 2.3.12. *For all* $\boldsymbol{v}_h \in \boldsymbol{H}^1(\mathcal{T}_h)$, *there exists a constant* $C_E > 0$ *independent of* h *such that*

$$C_E^2|||\boldsymbol{v}_h|||^2 \leq ||\nabla \boldsymbol{v}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{v}}_h]]||^2_{\mathscr{F}_h} \leq |||\boldsymbol{v}_h|||^2. \tag{2.42}$$

Theorem 2.3.13. *Let* $\boldsymbol{a}_h \in \boldsymbol{V}_h^0$ *and* $\boldsymbol{b}_h, \boldsymbol{c}_h \in \boldsymbol{W}_h^0$ *be fixed. Then, there exists a unique function* $\boldsymbol{\phi}_h \in \boldsymbol{V}_{bs}$ *such that*

$$\begin{cases} \boldsymbol{\pi}_0\boldsymbol{\phi}_h = \boldsymbol{a}_h, \\ \{\nabla\boldsymbol{\phi}_h\}|_F\boldsymbol{n}_F = \boldsymbol{b}_h|_F \ \forall F \in \mathscr{F}_h, \\ \{\bar{\boldsymbol{\phi}_h}\}|_F = \boldsymbol{c}_h|_F \ \forall F \in \mathcal{F}_i. \end{cases} \tag{2.43}$$

Moreover, $\boldsymbol{\phi}_h$ satisfies the following priori estimate

$$||\tilde{h}^{-\frac{1}{2}}\boldsymbol{\phi}_h||^2_{\mathcal{T}_h} + |||\boldsymbol{\phi}_h|||^2 \leq M_P(||\tilde{h}^{-1}\boldsymbol{a}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h||^2_{\mathscr{F}_h} + ||\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h||^2_{\mathcal{F}_i})$$

where M_P is a constant independent of h.

2.4 Analysis of Non-Symmetric DG Solution of Stokes Equations

In this section I prove the existence and uniqueness of the solution to the non-symmetric penalty-free method. A priori error estimation is also derived.

2.4.1 Existence of the Solution

In this section I use generalized Lax-Milgram theorem to prove the existence and uniqueness of the solution of the non-symmetric Stokes scheme. I first state the generalized Lax-Milgram theorem and then prove each step of the theorem for my scheme. I seek $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ that satisfies for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{bs} \times Q_h$

$$a(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) - b(q, \boldsymbol{u}_h) = F(\boldsymbol{v}, q). \tag{2.44}$$

I will use the generalized Lax-Milgram theorem and show that there exists a unique $(\boldsymbol{u}_h, p_h)$ satisfying (2.44). First let us state the Lax-Milgram theorem [22]:

Theorem 2.4.1 (Banach-Necas-Babuska). *Consider the following (abstract) problem*

$$\begin{aligned} &\text{Seek } \boldsymbol{u} \in \boldsymbol{W} \text{ such that} \\ &B(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{f}(\boldsymbol{v}),\ \forall \boldsymbol{v} \in \boldsymbol{V} \end{aligned} \tag{2.45}$$

- $\boldsymbol{W}$ *and* $\boldsymbol{V}$ *are vector spaces equipped with norms denoted by* $||\cdot||_{\boldsymbol{W}}$ *and* $||\cdot||_{\boldsymbol{V}}$, *respectively. In many applications,* $\boldsymbol{W}$ *and* $\boldsymbol{V}$ *are Hilbert spaces, but a more general case where* $\boldsymbol{V}$ *is a reflexive Banach space and* $\boldsymbol{W}$ *a Banach space can be considered. Unless stated otherwise, I henceforth assume that* $\boldsymbol{W}$ *and* $\boldsymbol{V}$ *are Banach spaces and that* $\boldsymbol{V}$ *is reflexive.* $\boldsymbol{W}$ *is called the solution space, and* $\boldsymbol{V}$ *is called the* test space.

- B *is a continuous bilinear form on* $\boldsymbol{W} \times \boldsymbol{V}$, *henceforth, I shall also say that a is bounded on* $\boldsymbol{W} \times \boldsymbol{V}$.

- $\boldsymbol{f}$ *is a continuous linear form on* $\boldsymbol{V}$, *i.e.,* $\boldsymbol{f} \in \boldsymbol{V}'$. *To simplify the notation, I write* $\boldsymbol{f}(\boldsymbol{v})$ *instead of* $< \boldsymbol{f}, \boldsymbol{v} >_{\boldsymbol{V}',\boldsymbol{V}}$.

then, problem (2.45) is well-posed (in the sense introduced by Hadamard [22]) if and only if

$$\exists \alpha > 0,\ \inf_{\boldsymbol{w}\in\boldsymbol{W}} \sup_{\boldsymbol{v}\in\boldsymbol{V}} \frac{B(\boldsymbol{w}, \boldsymbol{v})}{||\boldsymbol{w}||_{\boldsymbol{W}}||\boldsymbol{v}||_{\boldsymbol{V}}} \geq \alpha, \tag{2.46}$$

$$\forall \boldsymbol{v} \in \boldsymbol{V},\ (\forall \boldsymbol{w} \in \boldsymbol{W},\ B(\boldsymbol{w}, \boldsymbol{v}) = 0) \Rightarrow (\boldsymbol{v} = 0). \tag{2.47}$$

Moreover, the following a priori estimate holds

$$\forall \boldsymbol{f} \in \boldsymbol{V}',\ ||\boldsymbol{u}||_{\boldsymbol{W}} \leq \frac{1}{\alpha}||\boldsymbol{f}||_{\boldsymbol{V}'}. \tag{2.48}$$

I apply Theorem 2.4.1 with $\boldsymbol{W} = \boldsymbol{V} = \boldsymbol{V}_{bs} \times Q_h$. Since this is now a finite dimensional problem, the statement in (2.46) is equivalent to statement in (2.47) (see proposition (2.21) in [22]).

I define the bilinear form $S : (\boldsymbol{V}_{bs} \times Q_h) \times (\boldsymbol{V}_{bs} \times Q_h) \to \mathbb{R}$ by

$$S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)) = a(\boldsymbol{u}_h, \boldsymbol{v}) + b(p_h, \boldsymbol{v}) - b(q, \boldsymbol{u}_h). \tag{2.49}$$

Problem (2.44) becomes

$$S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)) = F(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_{bs} \times Q_h. \tag{2.50}$$

Proposition 2.4.2. *There exists a constant $\beta > 0$ independent of h such that for all $(\boldsymbol{u}_h, p_h)$ in $\boldsymbol{V}_{bs} \times Q_h$ there holds*

$$\beta |||(\boldsymbol{u}_h, p_h)||| \leq \sup_{(\boldsymbol{v},q) \in \boldsymbol{V}_{bs} \times Q_h} \frac{S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q))}{|||(\boldsymbol{v}, q)|||}. \tag{2.51}$$

The proof of this proposition is given at the end of this section. To apply the Lax-Milgram Theorem 2.4.1 I first show that the bilinear form S is continuous.

Proposition 2.4.3. *There is a constant M independent of h such that*

$$|S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q))| \leq M \; |||(\boldsymbol{u}_h, p_h)||| \; |||(\boldsymbol{v}, q)||| \quad \forall (\boldsymbol{u}_h, p_h), \;\; \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_{bs} \times Q_h. \tag{2.52}$$

Proof. From [9] there is a constant M_1 independent of h such that

$$|a(\boldsymbol{u}_h, \boldsymbol{v})| \leq M_1 |||(\boldsymbol{u}_h, p_h)||| \; |||(\boldsymbol{v}, q)|||. \tag{2.53}$$

By definition of b and using integration by parts (2.22) $b(p_h, \boldsymbol{v}) = \sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla p_h \cdot \boldsymbol{v} - \sum_{F \in \mathscr{F}} ([p_h], \{\boldsymbol{v}\})$ and therefore by Cauchy-Schwarz's inequality, trace inequality

(2.27) and (2.41) there exists a constant M_2 independent of h such that

$$|b(p_h, \boldsymbol{v})| \leq M_2 |||p_h|||_Q |||\boldsymbol{v}|||, \tag{2.54}$$

$$|b(q, \boldsymbol{v})| \leq M_2 |||q|||_Q |||\boldsymbol{v}|||. \tag{2.55}$$

Adding (2.53), (2.54) and (2.55) proves the continuity of the bilinear forms. □

Next, I show continuity of the right-hand side of (2.50). By Cauchy-Schwarz's inequality and (2.41) I have

$$|F(\boldsymbol{v}, q)| \leq ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)} ||\boldsymbol{v}||_{L^2(\Omega)} \leq M_4 ||\boldsymbol{f}||_{L^2(\Omega)} |||\boldsymbol{v}|||,$$

where M_4 is a constant independent of h. Therefore, by Lax-Milgram Theorem 2.4.1 there exists a unique $(\boldsymbol{u}_h, p_h)$ in $\boldsymbol{V}_{bs} \times Q_h$ satisfying (2.50).

Proof of the inf-sup condition:

Now, I break the proof of Proposition 2.4.2 in the following two steps:

- There exists a constant $M_1 > 0$ independent of h such that for each fixed pair $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ there exists a pair $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$ with

$$M_1 |||(\boldsymbol{u}_h, p_h)|||^2 \leq a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) \tag{2.56}$$

  and

- There exists a constant $M_2 > 0$ independent of h such that for the choice $\boldsymbol{v}_h, q_h$

in (2.56),

$$|||(\boldsymbol{v}_h, q_h)||| \leq M_2|||(\boldsymbol{u}_h, p_h)|||. \tag{2.57}$$

The proof of (2.56) and (2.57) follows closely the proof of Lemmas 5.2 and 5.3 in [9]. I fix $\boldsymbol{u}_h$ in $\boldsymbol{V}_{bs}$ and p_h in Q_h. Let $\boldsymbol{w}_h \in \boldsymbol{V}_{bs}$ be the projection defined in Theorem 2.3.13 with the arguments

$$\boldsymbol{a}_h = \boldsymbol{0} \ , \ \ \boldsymbol{b}_h = \widetilde{h}^{-1}[[\bar{\boldsymbol{u}_h}]]\boldsymbol{n}_F, \quad \text{and} \quad \boldsymbol{c}_h = \boldsymbol{0} \tag{2.58}$$

and let $\boldsymbol{z}_h$ be the projection defined in Theorem 2.3.13 with the arguments

$$\boldsymbol{a}_h = \tilde{h}^2\nabla p_h, \ \boldsymbol{b}_h = \boldsymbol{0} \ \textit{and} \ \boldsymbol{c}_h = -\tilde{h}[p_h]. \tag{2.59}$$

I compute:

$$a(\boldsymbol{u}_h, \boldsymbol{u}_h) = ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} \geq \frac{1}{2}||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h}. \tag{2.60}$$

Using (2.24), (2.19) , Corollary 2.3.5, Lemma 2.3.3 and $\boldsymbol{L}^2$ projection $\boldsymbol{\pi}_0$ on $\boldsymbol{V}^0_h$, (see (2.32)) I have,

$$\begin{aligned} a(\boldsymbol{u}_h, \boldsymbol{w}_h) &= -(\Delta \boldsymbol{u}_h, \boldsymbol{w}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} + ([[\boldsymbol{u}_h]]\boldsymbol{n}_F, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F)_{\mathscr{F}_h} \\ &= -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\bar{\boldsymbol{w}_h}\})_{\mathcal{F}_i} + ([[\bar{\boldsymbol{u}_h}]]\boldsymbol{n}_F, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F)_{\mathscr{F}_h}. \end{aligned} \tag{2.61}$$

From Theorem 2.3.13, $\boldsymbol{\pi}_0 \boldsymbol{w}_h = \boldsymbol{0}$, $\{\bar{\boldsymbol{w}_h}\}|_F = \boldsymbol{0}$ for all $F \in \mathcal{F}_i$ and $\{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F = \tilde{h}^{-1}[[\bar{\boldsymbol{u}_h}]]\boldsymbol{n}_F$ for all $F \in \mathscr{F}_h$. Thus, by (2.38)

$$a(\boldsymbol{u}_h, \boldsymbol{w}_h) = ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]\boldsymbol{n}_F||^2_{\mathscr{F}_h} = ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}.$$

Moreover, by integration by parts I obtain

$$
\begin{aligned}
b(p_h, \boldsymbol{w}_h) &= -(p_h, \nabla \cdot \boldsymbol{w}_h)_{\mathcal{T}_h} + (\{p_h\}, [\boldsymbol{w}_h])_{\mathscr{F}_h} \\
&= (\nabla p_h, \boldsymbol{w}_h)_{\mathcal{T}_h} - (\{p_h\}, [\boldsymbol{w}_h])_{\mathscr{F}_h} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} + (\{p_h\}, [\boldsymbol{w}_h])_{\mathscr{F}_h} \\
&= (\nabla p_h, \boldsymbol{w}_h)_{\mathcal{T}_h} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i}.
\end{aligned}
$$

Since $\nabla p_h \in \boldsymbol{V}_h^0$ and $\boldsymbol{\pi}_0 \boldsymbol{w}_h = \boldsymbol{0}$, I have,

$$
b(p_h, \boldsymbol{w}_h) = (\nabla p_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{T}_h} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} = -([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i}.
$$

Furthermore, since $p_h \in Q_h$ I write $p_h = p_{h,c} + p_{h,d}$ with $p_{h,c} \in Q_{h,c}^1$ and $p_{h,d} \in Q_{h,d}^0$. Using the fact that $\{\bar{\boldsymbol{w}}_h\} = 0$ on interior faces, I obtain

$$
\begin{aligned}
b(p_h, \boldsymbol{w}_h) &= -([p_{h,c}] + [p_{h,d}], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} = -([p_{h,d}], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} \\
&= -([p_{h,d}], \{\bar{\boldsymbol{w}}_h\})_{\mathcal{F}_i} = 0
\end{aligned}
$$

and

$$
b(p_h, \boldsymbol{z}_h) = (\nabla p_h, \boldsymbol{\pi}_0 \boldsymbol{z}_h)_{\mathcal{T}_h} - ([p_{h,d}], \{\bar{\boldsymbol{z}}_h\})_{\mathcal{F}_i} = ||\tilde{h} \nabla p_h||_{\mathcal{T}_h}^2 + ||\tilde{h}^{\frac{1}{2}} [p_h]||_{\mathcal{F}_i}^2 = |||p_h|||_Q^2.
$$

Now, by applying integration by parts, I obtain as in (2.61)

$$
a(\boldsymbol{u}_h, \boldsymbol{z}_h) = -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{z}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\bar{\boldsymbol{z}}_h\})_{\mathcal{F}_i} - ([[\bar{\boldsymbol{u}}_h]] \boldsymbol{n}_F, \{\nabla \boldsymbol{z}_h\} \boldsymbol{n}_F)_{\mathscr{F}_h}.
$$

By Theorem 2.3.13, I have $\boldsymbol{\pi}_0 \boldsymbol{z}_h = \tilde{h}^2 \nabla p_h, \{\bar{\boldsymbol{z}}_h\} = -\tilde{h}[p_h]$ and $\{\nabla \boldsymbol{z}_h\} \boldsymbol{n}_F = \boldsymbol{0}$. So, by inverse inequality (2.25), trace inequality (2.29), Cauchy-Schwarz's and Young's inequalities I obtain

$$
\begin{aligned}
a(\boldsymbol{u}_h, \boldsymbol{z}_h) &= -(\Delta \boldsymbol{u}_h, \tilde{h}^2 \nabla p_h)_{\mathcal{T}_h} - ([[\nabla \boldsymbol{u}_h]], \tilde{h}[p_h])_{\mathcal{F}_i}, \\
|a_{-1}(\boldsymbol{u}_h, \boldsymbol{v}_h)| &\leq ||\Delta \boldsymbol{u}_h||_{\mathcal{T}_h} \, ||\tilde{h}^2 \nabla p_h||_{\mathcal{T}_h} + ||[[\nabla \boldsymbol{u}_h]]||_{\mathcal{F}_i} \, ||\tilde{h}[p_h]||_{\mathcal{F}_i} \\
&\leq C_I ||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h} \, ||\tilde{h} \nabla p_h||_{\mathcal{T}_h} + C C_T ||\tilde{h}^{-\frac{1}{2}} \nabla \boldsymbol{u}_h||_{\mathcal{F}_i} \, ||\tilde{h}[p_h]||_{\mathcal{F}_i} \\
&\leq C ||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h} (||\tilde{h} \nabla p_h||_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[p_h]||_{\mathcal{F}_i}).
\end{aligned}
$$

So, I have

$$
a(\boldsymbol{u}_h, \boldsymbol{v}_h) \geq -C_2 ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - \frac{1}{2} |||p_h|||^2_Q
$$

where C_2 is a constant independent of h.

In summary I obtained,

$$
a(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \frac{1}{2} ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h}, \tag{2.62}
$$

$$
a(\boldsymbol{u}_h, \boldsymbol{w}_h) = ||\widetilde{h}^{-\frac{1}{2}} [[\bar{\boldsymbol{u}}_h]]||^2_{\mathscr{F}_h}, \tag{2.63}
$$

$$
b(p_h, \boldsymbol{w}_h) = 0, \tag{2.64}
$$

$$
b(p_h, \boldsymbol{z}_h) = |||p_h|||^2_Q, \tag{2.65}
$$

$$
a(\boldsymbol{u}_h, \boldsymbol{z}_h) \geq -C_2 ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - \frac{1}{2} |||p_h|||^2_Q, \tag{2.66}
$$

where C_2 is a constant independent of h.

To prove (2.56) I choose $\boldsymbol{v}_h = \boldsymbol{u}_h + \frac{1}{2}\boldsymbol{w}_h + \frac{1}{4C_2}\boldsymbol{z}_h$ and $q_h = p_h$

$$
\begin{aligned}
Y &\equiv a(\boldsymbol{u}_h, \boldsymbol{u}_h + \frac{1}{2}\boldsymbol{w}_h + \frac{1}{4C_2}\boldsymbol{z}_h) \\
&+ b(p_h, \boldsymbol{u}_h + \frac{1}{2}\boldsymbol{w}_h + \frac{1}{4C_2}\boldsymbol{z}_h) - b(q_h, \boldsymbol{u}_h) \\
&= a(\boldsymbol{u}_h, \boldsymbol{u}_h) + \frac{1}{2}a(\boldsymbol{u}_h, \boldsymbol{w}_h) + \frac{1}{4C_2}a(\boldsymbol{u}_h, \boldsymbol{z}_h) \\
&+ b(p_h, \boldsymbol{u}_h) + \frac{1}{2}b(p_h, \boldsymbol{w}_h) + \frac{1}{4C_2}b(p_h, \boldsymbol{z}_h) - b(p_h, \boldsymbol{u}_h)
\end{aligned}
$$

Taking $\boldsymbol{v}_h = \boldsymbol{0}$ and $q_h = p_h$ in (2.50) yields

$$
b(p_h, \boldsymbol{u}_h) = 0. \tag{2.67}
$$

Using (2.67) and (2.64) I obtain

$$
Y \geq a(\boldsymbol{u}_h, \boldsymbol{u}_h) + \frac{1}{2}a(\boldsymbol{u}_h, \boldsymbol{w}_h) + \frac{1}{4C_2}a(\boldsymbol{u}_h, \boldsymbol{z}_h) + \frac{1}{4C_2}b(p_h, \boldsymbol{z}_h). \tag{2.68}
$$

Now by plugging (2.62), (2.63), (2.65), and (2.66) in (2.68) I obtain

$$
\begin{aligned}
Y \geq{}& \frac{1}{2}||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + \frac{1}{2}||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||^2_{\mathscr{F}_h} - \frac{1}{4C_2}C_2||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} \\
&- \frac{1}{4C_2}\frac{1}{2}|||p_h|||^2_Q + \frac{1}{4C_2}|||p_h|||^2_Q.
\end{aligned} \tag{2.69}
$$

Thus by (2.42) I obtain

$$
Y \geq (\frac{1}{4} + \frac{M}{2})(||\nabla \boldsymbol{u}_h||^2) + \frac{M}{2}(||[\boldsymbol{u}_h]||^2) + \frac{1}{8C_2}(||p_h||^2_Q). \tag{2.70}
$$

By choosing $M_1 = \min\{\frac{M}{2}, \frac{1}{4} + \frac{M}{2}, \frac{1}{8C_2}\}$ I proved (2.56). I now prove (2.4.1). By a priori estimates in Theorem 2.3.13 there exists a constant M independent of h such that

$$|||\boldsymbol{w}_h|||^2 \leq M||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]\boldsymbol{n}_F||^2_{\mathscr{F}_h},$$
$$|||\boldsymbol{z}_h|||^2 \leq M|||p_h|||^2_Q. \tag{2.71}$$

Using (2.38) and (2.31) I have

$$|||\boldsymbol{w}_h|||^2 \leq M|||\boldsymbol{u}_h|||^2. \tag{2.72}$$

Therefore, using the definition of $\boldsymbol{v}_h$ I obtain

$$|||\boldsymbol{v}_h|||^2 \leq 4|||\boldsymbol{u}_h|||^2 + 0|||\boldsymbol{w}_h|||^2 + \frac{1}{C_2}|||\boldsymbol{z}_h|||^2$$
$$\leq M(|||\boldsymbol{u}_h|||^2 + |||p_h|||^2_Q).$$

And I clearly obtain (2.51) with a constant β independent of h.

2.4.2 A Priori Error Estimates

Define the following quantities

$$\boldsymbol{\eta}_u = \boldsymbol{u} - i_c\boldsymbol{u}, \quad \boldsymbol{\xi}_u = \boldsymbol{u}_h - i_c\boldsymbol{u}, \tag{2.73}$$

$$\eta_p = p - P_\alpha p, \quad \xi_p = p_h - P_\alpha p, \tag{2.74}$$

where the projection $P_\alpha : L^2(\Omega) \to Q_h$ is defined by $P_\alpha = (1-\alpha)\pi_0 + \alpha C_h$. By definition of $\boldsymbol{CR}$, I observe that $\boldsymbol{\xi}_u \in \boldsymbol{V}_{bs}$ and $\xi_p \in Q_h$. Furthermore, P_α satisfies the error estimate

$$||v - P_\alpha v||_{\mathcal{T}_h} + ||\tilde{h}\nabla(v - P_\alpha v)||_{\mathcal{T}_h} \leq ch^{\alpha\gamma+1}|v|_{\alpha\gamma+1,\mathcal{T}_h}$$

for all $v \in H^{\gamma+1}(\mathcal{T}_h), \gamma \in \{0,1\}$.

Theorem 2.4.4. *Let $\boldsymbol{\eta}_u \in \boldsymbol{H}^2(\Omega) + \boldsymbol{CR}$ and $\eta_p \in H^{\gamma+1}(\Omega) + Q_h$, with $\gamma \in \{0,1\}$ and $Q_h = \alpha Q^1_{h,c} \oplus \beta Q^0_{h,d}$ for $(\alpha, \beta) \in \{(1,0),(0,1),(1,1)\}$, be defined as (2.73) and (2.74). Then there exists a constant $M > 0$ independent of h such that*

$$|||\boldsymbol{\eta}_u, \eta_p||| \leq Mh|\boldsymbol{u}|_{2,\mathcal{T}_h} + Mh^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{T}_h}. \tag{2.75}$$

Lemma 2.4.5. *There exists a constant $\beta > 0$ independent of h such that there holds $\forall(\boldsymbol{\xi}_u, \xi_p) \in \boldsymbol{V}_{bs} \times Q_h$*

$$\beta|||(\boldsymbol{\xi}_u, \xi_p)||| \leq \sup \frac{a(\boldsymbol{\xi}_u, \boldsymbol{v}) + b(\xi_p, \boldsymbol{v}) - b(q, \boldsymbol{\xi}_u)}{|||(\boldsymbol{v}, q)|||}.$$

Proof. Since $(\boldsymbol{\xi}_u, \xi_p)$ is in $\boldsymbol{V}_{bs} \times Q_h$ then proof is as the proof of Proposition 2.4.2. $\square$

Theorem 2.4.6. *Let $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega), p \in H^{\gamma+1}(\Omega), \gamma \in \{0,1\}$, be the exact solution of the problem (2.1) and let $\boldsymbol{u}_h \in \boldsymbol{V}_{bs}, p_h \in Q_h$, be the approximation defined by (2.44). Then, there exists a constant $M > 0$ independent of h such that*

$$|||\boldsymbol{u} - \boldsymbol{u}_h||| + |||p - p_h|||_Q + ||p - p_h|| \leq Mh|\boldsymbol{u}|_{2,\mathcal{T}_h} + Mh^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{T}_h}. \tag{2.76}$$

Proof. Proof is given in [9]. I remind it here.

Let us first establish the a priori estimate for the triple norm $|||\cdot|||$. Split the error in a standard manner into two parts

$$|||(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h)||| \leq |||(\boldsymbol{\eta}_u, \eta_p)||| + |||(\boldsymbol{\xi}_u, \xi_p)||| \tag{2.77}$$

by Theorem 2.4.4, it follows that

$$|||\boldsymbol{\eta}_u, \eta_p||| \leq Mh|\boldsymbol{u}|_{2,\mathcal{T}_h} + Mh^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{T}_h}.$$

So I need estimation for the second term of (2.78). Using consistency and subtracting equation (2.44) with the exact solution from (2.44), I obtain the following error equation

$$a(\boldsymbol{\xi}_u, \boldsymbol{v}_h) + b(\xi_p, \boldsymbol{v}_h) - b(q_h, \boldsymbol{\xi}_u) = a(\boldsymbol{\eta}_u, \boldsymbol{v}_h) + b(\eta_p, \boldsymbol{v}_h) - b(q_h, \boldsymbol{\eta}_u) \tag{2.78}$$

By Lemma 2.4.5, continuity of S and (2.78)

$$\begin{aligned} |||\boldsymbol{\xi}_u, \xi_p||| &\leq \beta \sup_{(\boldsymbol{v}_h, q_h)\in \boldsymbol{V}_{bs}\times Q_h} \frac{a(\boldsymbol{\xi}_u, \boldsymbol{v}_h) + b(\xi_p, \boldsymbol{v}_h) - b(q_h, \boldsymbol{\xi}_u)}{|||(\boldsymbol{v}_h, q_h)} \\ &= \beta \sup_{(\boldsymbol{v}_h, q_h)\in \boldsymbol{V}_{bs}\times Q_h} \frac{a(\boldsymbol{\eta}_u, \boldsymbol{v}_h) + b(\eta_p, \boldsymbol{v}_h) - b(q_h, \boldsymbol{\eta}_u)}{|||(\boldsymbol{v}_h, q_h)} \\ &\leq Mh|\boldsymbol{u}|_{2,\mathcal{T}_h} + Mh^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{T}_h}. \end{aligned}$$

Therefore, I obtain

$$|||\boldsymbol{u}-\boldsymbol{u}_h||| + |||p-p_h|||_Q \leq Mh|\boldsymbol{u}|_{2,\mathcal{T}_h} + Mh^{1+\alpha\gamma}|p|_{1+\alpha\gamma,\mathcal{T}_h}. \tag{2.79}$$

To prove the error estimate for $||p-p_h||_{\mathcal{T}_h}$ I have ([26]) that there exists $\boldsymbol{v}_p \in \boldsymbol{H}_0^1(\Omega)$ such that $\nabla \cdot \boldsymbol{v}_p = p - p_h$ and $||\nabla \boldsymbol{v}_p|| \leq M||p-p_h||_{\mathcal{T}_h}$, where $M > 0$ is a constant independent of h. I write $||p-p_h||^2_{L^2(\mathcal{T}_h)} = T_1 + T_2$ where

$$T_1 = -(\nabla(p-p_h), \boldsymbol{v}_p - \boldsymbol{C}_h\boldsymbol{v}_p)_{\mathcal{T}_h} + ([p-p_h], \{\boldsymbol{v}_p - \boldsymbol{C}_h\boldsymbol{v}_p\})_{\mathcal{F}_i} +$$

$$T_2 = -\,(\nabla(p-p_h), \boldsymbol{C}_h\boldsymbol{v}_p)_{\mathcal{T}_h} + ([p-p_h], \{\boldsymbol{C}_h\boldsymbol{v}_p\})_{\mathcal{F}_i}$$

and $\boldsymbol{C}_h$ is vectorial Clement interpolation operator.

Using Cauchy-Schwarz's inequality, the trace inequality (2.29), (2.36) and consistency of form I obtain

$$\begin{aligned} T_1 &\leq ||\nabla(p-p_h)||_{\mathcal{T}_h}\, ||\boldsymbol{v}_p - \boldsymbol{C}_h\boldsymbol{v}_p||_{\mathcal{T}_h} + ||[p-p_h]||_{\mathscr{F}_h}\, ||\{\boldsymbol{v}_p - \boldsymbol{C}_h\boldsymbol{v}_p\}||_{\mathscr{F}_h} \\ &\leq M\tilde{h}||\nabla(p-p_h)||_{\mathscr{F}_h}\, ||\nabla\boldsymbol{v}_p||_{\mathcal{T}_h} + M\tilde{h}^{\frac{1}{2}}||[p-p_h]||_{\mathscr{F}_h}\, ||\nabla\boldsymbol{v}_p||_{\mathcal{T}_h} \\ &\leq M(||\tilde{h}\nabla(p-p_h)||_{\mathscr{F}_h} + ||\tilde{h}^{\frac{1}{2}}[p-p_h]||_{\mathscr{F}_h})||\nabla\boldsymbol{v}_p|| \\ &\leq M|||p-p_h|||_Q||p-p_h||_{\mathcal{T}_h} \end{aligned} \tag{2.80}$$

and

$$\begin{aligned} T_2 &= -b(p-p_h, \boldsymbol{C}_h\boldsymbol{v}_p) = a(\boldsymbol{u}-\boldsymbol{u}_h, \boldsymbol{C}_h\boldsymbol{v}_p) \\ &= (\nabla(\boldsymbol{u}-\boldsymbol{u}_h), \nabla\boldsymbol{C}_h\boldsymbol{v}_p)_{\mathcal{T}_h} + ([[\boldsymbol{u}-\boldsymbol{u}_h]], \{\nabla\boldsymbol{C}_h\boldsymbol{v}_p\})_{\mathscr{F}_h} \\ &\leq (||\nabla(\boldsymbol{u}-\boldsymbol{u}_h)||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\boldsymbol{u}-\boldsymbol{u}_h||^2_{\mathscr{F}_h})^{\frac{1}{2}}(||\nabla\boldsymbol{C}_h\boldsymbol{v}_p||^2_{\mathcal{T}_h} + ||\tilde{h}^{\frac{1}{2}}\{\nabla\boldsymbol{C}_h\boldsymbol{v}_p\}||^2_{\mathscr{F}_h})^{\frac{1}{2}} \\ &\leq C(||\nabla(\boldsymbol{u}-\boldsymbol{u}_h)||_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\boldsymbol{u}-\boldsymbol{u}_h]]||_{\mathscr{F}_h})||p-p_h||_{\mathcal{T}_h}. \end{aligned} \tag{2.81}$$

By combining (2.80) and (2.81) I obtain,

$$||p-p_h|| \leq C(|||\boldsymbol{u}-\boldsymbol{u}_h||| + |||p-p_h|||_Q) \tag{2.82}$$

where $C > 0$ is a constant independent of h. By (2.79) and (2.82), I obtain (2.76). $\square$

Chapter 3

Navier-Stokes Equations

In this chapter I formulate the penalty-free symmetric and non-symmetric schemes for the incompressible Navier-Stokes equations. I restate the solution of the discrete problem as a fixed point of a problem-related map, that yields an Oseen problem. I show the map is well posed using the generalized Lax-Milgram theorem. I apply the Leray-Schauder fixed point theorem to show that the map has a fixed point. Optimal a priori error estimates are derived. I finish the chapter by presenting several numerical examples. To the best of my knowledge this chapter is the first work for the penalty-free symmetric and non-symmetric scheme for the incompressible Navier-Stokes equations. The structure of the chapter is as follows:

I first state the model problem. In section 3.1.1 I give the numerical scheme for the symmetric and non-symmetric penalty-free incompressible Navier-Stokes equations. In section 3.2 existence of the numerical solution is discussed. In this chapter I first express the numerical solution of the scheme as a fixed point of the problem-related map called G. From there the proof of the existence of the numerical solution follows in two parts. First, I show that the map G is well-defined. Proving the generalized Lax-Milgram theorem is my tool to show the map G is well-defined. Since the proof of the inf-sup condition in Lax-Milgram theorem is more involved, I give the proof of the inf-sup condition in section 3.3. Second, I show the map G has a fixed point for which I apply Leray-Schauder fixed-point theorem to my map. The proof of the continuity of map G is more involved and therefore, I allocate section 3.4 to it. In section 3.5 I prove the error estimates. Finally in section 3.6 I modify Dr Riviere discontinuous Galerkin codes for the Navier-Stokes equations to include bubble basis and show some numerical examples. I do not use a preconditioner and the modified parts of the code by me is not optimal. The goal of this chapter is to highlight the results by showing some numerical examples.

3.1 Model Problem and Scheme

I define the Navier-Stokes equations on a convex polygon $\Omega \in \mathbb{R}^d$ with $(d = 2, 3)$. Given $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$, such that

$$-\Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{u} = 0, \quad \text{on } \partial\Omega. \tag{3.3}$$

The Sobolev spaces above are defined in (2.2) and (2.3).

3.1.1 Numerical Solution

The diffusive term of the Navier-Stokes equations is discretized by the following bilinear form

$$a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_{\mathcal{T}_h} - (\{\nabla \boldsymbol{u}_h\}, [[\boldsymbol{v}_h]])_{\mathscr{F}_h} + \epsilon([[\boldsymbol{u}_h]], \{\nabla \boldsymbol{v}_h\})_{\mathscr{F}_h} \tag{3.4}$$

where ϵ takes the values $+1$ or -1. One can check that

$$a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}_h) = (\nabla \boldsymbol{u}_h, \nabla \boldsymbol{v}_h)_{\mathcal{T}_h} - (\{\nabla \boldsymbol{u}_h\}\boldsymbol{n}_\tau, j(\boldsymbol{v}_h))_{\mathscr{F}_h} + \epsilon(j(\boldsymbol{u}_h), \{\nabla \boldsymbol{v}_h\}\boldsymbol{n}_\tau)_{\mathscr{F}_h},$$

which is the standard form given in [26]. The pressure term is discretized by (2.16). To discretize the nonlinear convection term I introduce the notation z^{int} respectively z^{ext} to denote the restriction of z to the element τ, respectively the neighboring element to τ. The vector $\boldsymbol{n}_\tau$ is the normal outward to τ.

$$\begin{aligned} c(\boldsymbol{w}_h, \boldsymbol{v}_h; \boldsymbol{z}_h, \boldsymbol{\theta}_h) = & \sum_{\tau \in \mathcal{T}_h} \int_\tau (\boldsymbol{v}_h \cdot \nabla \boldsymbol{z}_h) \cdot \boldsymbol{\theta} + \frac{1}{2} \sum_{\tau \in \mathcal{T}_h} \int_\tau (\nabla \cdot \boldsymbol{v}_h) \boldsymbol{z}_h \cdot \boldsymbol{\theta}_h \\ & + \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau_-^{\boldsymbol{w}_h}} |\{\boldsymbol{v}_h\} \cdot \boldsymbol{n}_\tau| (\boldsymbol{z}_h^{int} - \boldsymbol{z}_h^{ext}) \cdot \boldsymbol{\theta}_h^{int} - \frac{1}{2} \sum_{F \in \mathscr{F}_h} \int_F [\boldsymbol{v}_h] \{\boldsymbol{z}_h \cdot \boldsymbol{\theta}\} \end{aligned} \tag{3.5}$$

where

$$\partial \tau_-^{\boldsymbol{w}_h} = \{\boldsymbol{x} \in \partial \tau \;:\; \{\boldsymbol{w}_h\} \cdot \boldsymbol{n}_\tau < 0\}. \tag{3.6}$$

The form c is well-studied in the literature, in particular in [26, 11, 40]. I recall the following continuity bound proved in Proposition 4.1 of [26].

Lemma 3.1.1. *There is a constant C_0 independent of h such that*

$$\forall \boldsymbol{z}_h, \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_{bs} \quad |c(\boldsymbol{z}_h, \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h)| \leq C_0 |||\boldsymbol{u}_h|||\, |||\boldsymbol{v}_h|||\, |||\boldsymbol{w}_h|||. \tag{3.7}$$

The following positivity result is proved in (6.6) in [26]

Lemma 3.1.2.

$$c(\boldsymbol{v}_h, \boldsymbol{v}_h, \boldsymbol{w}_h, \boldsymbol{w}_h) \geq 0 \quad \forall \boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{V}_{bs} \tag{3.8}$$

The numerical scheme is: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ such that

$$a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) + c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = F(\boldsymbol{v}_h, q_h) \tag{3.9}$$

for all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$.

3.2 Existence of Solution to the Navier-Stokes Discretized Problem

In this part we establish and prove the existence of the numerical solution. I consider two cases: (i) penalty free non-symmetric and, (ii) penalty free symmetric and in both cases we define the solution of the scheme as a fixed point of a special map. I use the Leray-Schauder theorem to prove existence of a fixed point. I define a map G as follows:

$$G : \boldsymbol{V}_{bs} \times Q_h \to \boldsymbol{V}_{bs} \times Q_h$$
$$(\tilde{\boldsymbol{u}}_h, \tilde{p}_h) \mapsto (\boldsymbol{u}_h, p_h)$$

where $(\boldsymbol{u}_h, p_h)$ satisfies for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_{bs} \times Q_h$

$$a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) - b(q, \boldsymbol{u}_h) + c(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{u}}_h, \boldsymbol{u}_h, \boldsymbol{v}) = F(\boldsymbol{v}, q). \tag{3.10}$$

Observe that the fixed point of the map G is the solution of (3.9). I will use the Leray-Schauder theorem to prove existence of a fixed-point of G. I first state the Leray-Schauder theorem.

Theorem 3.2.1. *Let G be a continuous and compact mapping of a Banach space X into itself, such that the set*

$$\{x \in X : x = \lambda G x \textit{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then G has a fixed point.

The first step is to show that the map G is well-defined. So, we will use the generalized Lax-Milgram Theorem 2.4.1 and show that there exists a unique $(\boldsymbol{u}_h, p_h)$

satisfying (3.10).

I apply Theorem 2.4.1 to the following space $\boldsymbol{W} = \boldsymbol{V} = \boldsymbol{V}_{bs} \times Q_h$. Since this is now a finite dimensional problem, the statement in (2.46) is equivalent to statement in (2.47) (see proposition (2.21) in [22]). I define the bilinear form $S : (\boldsymbol{V}_{bs} \times Q_h) \times (\boldsymbol{V}_{bs} \times Q_h) \to \mathbb{R}$ by

$$S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)) = a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}) + b(p_h, \boldsymbol{v}) - b(q, \boldsymbol{u}_h) + c(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{u}}_h, \boldsymbol{u}_h, \boldsymbol{v}). \tag{3.11}$$

Problem (3.10) becomes

$$S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q)) = F(\boldsymbol{v}, q) \quad \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_{bs} \times Q_h. \tag{3.12}$$

To apply the Lax-Milgram Theorem 2.4.1 I first show that the bilinear form S is continuous.

Proposition 3.2.2. *There is a constant M independent of h such that*

$$|S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}, q))| \leq M \; |||(\boldsymbol{u}_h, p_h)||| \; |||(\boldsymbol{v}, q)||| \quad \forall (\boldsymbol{u}_h, p_h), \;\; \forall (\boldsymbol{v}, q) \in \boldsymbol{V}_{bs} \times Q_h. \tag{3.13}$$

Proof. From [9] there is a constant M_1 independent of h such that

$$|a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v})| \leq M_1 |||(\boldsymbol{u}_h, p_h)||| \; |||(\boldsymbol{v}, q)|||. \tag{3.14}$$

By definition of b and using integration by parts (2.22)

$$b(p_h, \boldsymbol{v}_h) = \sum_{\tau \in \mathcal{T}_h} \int_\tau \nabla p_h \cdot \boldsymbol{v}_h - \sum_{F \in \mathscr{F}} ([p_h], \{\boldsymbol{v}_h\}),$$

and therefore by Cauchy-Schwarz's inequality, trace inequality (2.27) and (2.41) there exists a constant M independent of h such that

$$|b(p_h, \boldsymbol{v}_h)| \leq M|||p_h|||_Q|||\boldsymbol{v}_h|||, \tag{3.15}$$

$$|b(q, \boldsymbol{v}_h)| \leq M|||q|||_Q|||\boldsymbol{v}_h|||. \tag{3.16}$$

Also, from Lemma 3.1.1, there is a constant $C_0 > 0$ independent of h such that

$$|c(\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{u}}_h, \boldsymbol{u}_h, \boldsymbol{v})| \leq C_0|||\widetilde{\boldsymbol{u}}_h|||\ |||\boldsymbol{u}_h|||\ |||\boldsymbol{v}||| \leq N|||(\boldsymbol{v}, q)|||\ |||(\boldsymbol{u}_h, p_h)||| \tag{3.17}$$

where $N = C_0|||\widetilde{\boldsymbol{u}}_h|||$. Adding (3.14), (3.15), (3.16) and (3.17) proves the continuity of the bilinear forms. □

Next, I show continuity of the right-hand side of (3.12). By Cauchy-Schwarz's inequality and (2.41) I have

$$\begin{aligned} |F(\boldsymbol{v}, q)| &\leq ||\boldsymbol{f}||_{\boldsymbol{L}^2(\Omega)}||\boldsymbol{v}||_{L^2(\Omega)} \\ &\leq C_P||\boldsymbol{f}||_{L^2(\Omega)}|||\boldsymbol{v}|||. \end{aligned}$$

Finally the Lax-Milgram theorem requires an inf-sup condition stated below. The proof is given in Section 3.3.

Proposition 3.2.3. *Assume $\widetilde{\boldsymbol{u}}_h$ satisfies the following bound*

$$|||\widetilde{\boldsymbol{u}}_h||| \leq \mathcal{C} \tag{3.18}$$

where for the non-symmetric case ($\epsilon = 1$) the constant $\mathcal{C}$ is equal to $\frac{C_E}{2C_0M_P}$; and for the symmetric case ($\epsilon = -1$) the constant $\mathcal{C}$ is equal to $\frac{C_E}{2C_0(1+4C_T^2)M_P}$. Here C_E is the constant in (2.42), C_0 is the constant in (3.7) and C_T is the constant in (2.27).

There exists a constant $\beta > 0$ independent of h and of $\widetilde{\boldsymbol{u}}_h$ such that for all $(\boldsymbol{u}_h, p_h)$ in $\boldsymbol{V}_{bs} \times Q_h$ there holds

$$\beta |||(\boldsymbol{u}_h, p_h)||| \leq \sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_{bs}\times Q_h} \frac{a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}) + b(p_h, \boldsymbol{v}) - b(q, \boldsymbol{u}_h) + c(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{u}}_h, \boldsymbol{u}_h, \boldsymbol{v})}{|||(\boldsymbol{v}, q)|||}. \quad (3.19)$$

Therefore, by Lax-Milgram Theorem 2.4.1, under the condition (3.18) there exists a unique $(\boldsymbol{u}_h, p_h)$ in $\boldsymbol{V}_{bs} \times Q_h$ satisfying (3.12). In addition I have the following bound for $(\boldsymbol{u}_h, p_h)$

$$|||(\boldsymbol{u}_h, p_h)||| \leq \frac{C_P}{\beta} \|\boldsymbol{f}\|_{L^2(\Omega)},$$

where C_P is the Poincaré's constant in (2.40). Under the small data assumption

$$\|\boldsymbol{f}\|_{L^2(\Omega)} \leq \frac{\beta \mathcal{C}}{C_P} \quad (3.20)$$

I conclude that the solution $\boldsymbol{u}_h$ also satisfies the bound

$$|||\boldsymbol{u}_h||| \leq \mathcal{C}.$$

Define the space

$$\boldsymbol{X}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_{bs} : \quad |||\boldsymbol{v}_h||| \leq \mathcal{C}\}.$$

Thus I showed that the restriction of the map G to the space $\boldsymbol{X}_h \times Q_h$ is a well-defined map onto $\boldsymbol{X}_h \times Q_h$. The second step of the Leray-Schauder theorem is to show that the map G is continuous. The proof is given in Section 3.4.

Proposition 3.2.4. *Let $(\widetilde{\boldsymbol{u}_n})_{n\geq 0}$ be a sequence in $\boldsymbol{V}_{bs}$ that converges to $\tilde{\boldsymbol{u}}$ in $\boldsymbol{V}_{bs}$. Let $(\widetilde{p_n})_{n\geq 0}$ be a sequence in Q_h that converges to $\tilde{p}$ in Q_h. By definition set $(\boldsymbol{u}_n, p_n) = G(\widetilde{\boldsymbol{u}_n}, \widetilde{p_n})$ and $(\boldsymbol{u}, p) = G(\widetilde{\boldsymbol{u}}, \widetilde{p})$. Then, the sequence $(\boldsymbol{u}_n)_{n\geq 0}$ converges to $\boldsymbol{u}$ in $\boldsymbol{V}_{bs}$ and the sequence $(p_n)_{n\geq 0}$ converges to p in Q_h.*

Finally I need to show that the map G is compact. Since $\boldsymbol{V}_{bs} \times Q_h$ is a finite dimensional space, it is equivalent to show that G is sequentially compact. For sequentially compactness it is sufficient to show that G maps bounded sets to bounded sets.

Proposition 3.2.5. *Assume that $(\widetilde{\boldsymbol{u}_h}, \widetilde{p_h})$ is bounded by a constant then $(\boldsymbol{u}_h, p_h)$ is also bounded.*

Proof. From the inf-sup condition, I can write

$$\begin{aligned}
\beta|||(\boldsymbol{u}_h, p_h)||| &\leq \sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_{bs}\times Q_h} \frac{a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}) + b(\boldsymbol{v}, p_h) - b(q, \boldsymbol{u}_h) + c(\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{u}}_h, \boldsymbol{u}_h, \boldsymbol{v})}{|||(\boldsymbol{v}, q)|||} \\
&\leq \sup_{(\boldsymbol{v},q)\in \boldsymbol{V}_{bs}\times Q_h} \frac{F(\boldsymbol{v}, q)}{|||(\boldsymbol{v}, q)|||} \\
&\leq C_P \|\boldsymbol{f}\|_{L^2(\Omega)}.
\end{aligned}$$

Since the inf-sup constant β is independent of h and $\widetilde{\boldsymbol{u}}_h$ I can conclude. □

The last assumption in the Leray-Schauder theorem is to show that the set

$$Z = \{(\widetilde{\boldsymbol{u}_h}, \widetilde{p_h}) \in \boldsymbol{X}_h \times Q_h : \ (\widetilde{\boldsymbol{u}_h}, \widetilde{p_h}) = \lambda G(\widetilde{\boldsymbol{u}_h}, \widetilde{p_h}) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded.

Proof. Fix $0 \leq \lambda \leq 1$. Define $(\boldsymbol{u}_h, p_h) = G(\widetilde{\boldsymbol{u}}_h, \tilde{p}_h)$. From the inf-sup condition, I have

$$|||(\boldsymbol{u}_h, p_h)||| \leq \frac{C_P}{\beta} \|\boldsymbol{f}\|_{L^2(\Omega)}.$$

Since $(\widetilde{\boldsymbol{u}}_h, \tilde{p}_h) = \lambda(\boldsymbol{u}_h, p_h)$, this implies

$$|||(\widetilde{\boldsymbol{u}}_h, \tilde{p}_h||| \leq \lambda \frac{C_P}{\beta} \|\boldsymbol{f}\|_{L^2(\Omega)}$$

which is clearly bounded. □

Therefore I have shown that there exists a fixed point of the map G restricted to $\boldsymbol{X}_h \times Q_h$. I summarize the result in the following theorem.

Theorem 3.2.6. *Under the small data assumption (3.20) there exists $(\boldsymbol{u}_h, p_h) \in \boldsymbol{X}_h \times Q_h$ that satisfies*

$$a_\epsilon(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) + c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) = F(\boldsymbol{v}_h, q_h) \quad (\boldsymbol{v}, q) \in \boldsymbol{V_{bs}} \times Q_h. \tag{3.21}$$

3.3 Proof of the Inf-Sup Condition

In order to show that the map G defined in previous chapter is well-defined I need to show that (3.10) has a unique solution. To do so I use generalized Lax-Milgram. I proved F, S are continuous and now I have to prove that the inf-sup condition is satisfied. In this section I prove the inf-sup condition for both non-symmetric and symmetric cases.

3.3.1 Non-Symmetric Case

I first prove the following lemma that is used in the proof of the inf-sup condition.

Lemma 3.3.1. *Let $\boldsymbol{w}_h \in \boldsymbol{V}_{bs}$ be the projection defined in Theorem 2.3.13 with the arguments*

$$\boldsymbol{a}_h = \boldsymbol{0}\ ,\ \ \boldsymbol{b}_h = \tilde{h}^{-1}[[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F, \quad \textit{and} \quad \boldsymbol{c}_h = \boldsymbol{0} \tag{3.22}$$

and let $\boldsymbol{z}_h$ be the projection defined in Theorem 2.3.13 with the arguments

$$\boldsymbol{a}_h = \tilde{h}^2 \nabla p_h,\ \boldsymbol{b}_h = \boldsymbol{0} \textit{ and } \boldsymbol{c}_h = -\tilde{h}[p_h] \tag{3.23}$$

then,

$$c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{w}_h) \geq -N_0|||\widetilde{\boldsymbol{u}_h}|||(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}) \tag{3.24}$$

and

$$c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{z}_h) \geq -\frac{N_1}{2}|||\widetilde{\boldsymbol{u}_h}|||^2(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}) - \frac{1}{2}|||p_h|||^2_Q. \tag{3.25}$$

where N_0 and N_1 are constants independent of h. In fact, $N_0 = M_P C_0/C_E$ where C_E is the constant in (2.42) and C_0 is the constant in (3.7).

I recall that the definition of $[[\boldsymbol{u}_h]]$ is given in (2.6) and that $[[\bar{\boldsymbol{u}_h}]]$ is the L^2 projection on piecewise constants (see (2.30)).

Proof. By Theorem 2.3.13, (2.31) and Lemma 2.3.1 I have,

$$\begin{aligned} |||\boldsymbol{w}_h||| &\leq M_P||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]\boldsymbol{n}_F||_{\mathscr{F}_h} \\ &\leq M_P||\tilde{h}^{-\frac{1}{2}}[[\boldsymbol{u}_h]]\boldsymbol{n}_F||_{\mathscr{F}_h} \leq M_P||\tilde{h}^{-\frac{1}{2}}[[\boldsymbol{u}_h]]||_{\mathscr{F}_h} \\ &\leq M_P|||\boldsymbol{u}_h|||. \end{aligned} \tag{3.26}$$

I also have by Theorem 2.3.13

$$|||\boldsymbol{z}_h||| \leq M(||\tilde{h}\nabla p_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{\frac{1}{2}}[p]||_{\mathcal{F}_i})^{\frac{1}{2}} \leq M|||p_h|||_Q. \tag{3.27}$$

Using Lemma 3.1.1, (3.26) and (2.42), I have

$$\begin{aligned} |c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{w}_h)| &\leq C_0|||\widetilde{\boldsymbol{u}_h}|||\ |||\boldsymbol{u}_h|||\ |||\boldsymbol{w}_h||| \\ &\leq \frac{C_0}{C_E}|||\widetilde{\boldsymbol{u}_h}|||(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}). \end{aligned}$$

I also have from Lemma 3.1.1 and (3.27),

$$
\begin{aligned}
|c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{z}_h)| &\leq C_0|||\widetilde{\boldsymbol{u}_h}|||\ |||\boldsymbol{u}_h|||\ |||\boldsymbol{z}_h||| \\
&\leq N_1^{1/2}|||\widetilde{\boldsymbol{u}_h}|||\ (||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h})^{1/2}|||p_h|||_Q \\
&\leq \frac{N_1|||\widetilde{\boldsymbol{u}_h}|||^2}{2}(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}) + \frac{1}{2}|||p_h|||^2_Q.
\end{aligned}
$$

□

Assume now that $\widetilde{\boldsymbol{u}}_h \in \boldsymbol{V}_{bs}$ satisfies (3.18). This is equivalent to write

$$
|||\widetilde{\boldsymbol{u}}_h||| \leq \frac{1}{2N_0} \tag{3.28}
$$

where N_0 is the constant given in (3.24). Next, I break the proof of Proposition 3.2.3 in the following two parts: there exist constants M_3 and M_4 independent of h and $\widetilde{\boldsymbol{u}}_h$ such that for each fixed pair $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ there exists a pair $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$ with

$$
M_3|||(\boldsymbol{u}_h, p_h)|||^2 \leq a_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) + c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{v}_h) \tag{3.29}
$$

and

$$
|||(\boldsymbol{v}_h, q_h)||| \leq M_4|||(\boldsymbol{u}_h, p_h)|||. \tag{3.30}
$$

The proof of (3.29) and (3.30) follows the proof of Lemmas 5.2 and 5.3 in [9]. I fix $\boldsymbol{u}_h$ in $\boldsymbol{V}_{bs}$ and p_h in Q_h. Let $\boldsymbol{w}_h, \boldsymbol{z}_h$ be defined as Lemma 3.3.1.

$$
a_1(\boldsymbol{u}_h, \boldsymbol{u}_h) = ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h}. \tag{3.31}
$$

Using (2.24), (2.19), Corollary 2.3.5, Lemma 2.3.3 and $\boldsymbol{L}^2$ projection $\boldsymbol{\pi}_0$ on $\boldsymbol{V}_h^0$, (see

(2.32)) I have,

$$\begin{aligned} a_1(\boldsymbol{u}_h, \boldsymbol{w}_h) &= -(\Delta \boldsymbol{u}_h, \boldsymbol{w}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} + ([[\boldsymbol{u}_h]]\boldsymbol{n}_F, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F)_{\mathscr{F}_h} \\ &= -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\bar{\boldsymbol{w}}_h\})_{\mathcal{F}_i} + ([[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F)_{\mathscr{F}_h}. \end{aligned} \tag{3.32}$$

From Theorem 2.3.13 I have, $\boldsymbol{\pi}_0 \boldsymbol{w}_h = \boldsymbol{0}$, $\{\bar{\boldsymbol{w}}_h\}|_F = \boldsymbol{0}$ for all $F \in \mathcal{F}_i$ and $\{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F = \tilde{h}^{-1}[[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F$ for all $F \in \mathscr{F}_h$. Thus, by (2.38)

$$a_1(\boldsymbol{u}_h, \boldsymbol{w}_h) = ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F||^2_{\mathscr{F}_h} = ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||^2_{\mathscr{F}_h}.$$

Moreover, by integration by parts I obtain

$$\begin{aligned} b(p_h, \boldsymbol{w}_h) &= -(p_h, \nabla \cdot \boldsymbol{w}_h)_{\mathcal{T}_h} + (\{p_h\}, [\boldsymbol{w}_h])_{\mathscr{F}_h} \\ &= (\nabla p_h, \boldsymbol{w}_h)_{\mathcal{T}_h} - (\{p_h\}, [\boldsymbol{w}_h])_{\mathscr{F}_h} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} + (\{p_h\}, [\boldsymbol{w}_h])_{\mathscr{F}_h} \\ &= (\nabla p_h, \boldsymbol{w}_h)_{\mathcal{T}_h} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i}. \end{aligned}$$

Since $\nabla p_h \in \boldsymbol{V}_h^0$ and $\boldsymbol{\pi}_0 \boldsymbol{w}_h = \boldsymbol{0}$, I have,

$$b(p_h, \boldsymbol{w}_h) = (\nabla p_h, \boldsymbol{\pi}_0 \boldsymbol{w}_h)_{\mathcal{T}_h} - ([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} = -([p_h], \{\boldsymbol{w}_h\})_{\mathcal{F}_i}.$$

Furthermore, since $p_h \in Q_h$ I write $p_h = p_{h,c} + p_{h,d}$ with $p_{h,c} \in Q^1_{h,c}$ and $p_{h,d} \in Q^0_{h,d}$. Using the fact that $\{\bar{\boldsymbol{w}}_h\} = 0$ on interior faces, I obtain

$$\begin{aligned} b(p_h, \boldsymbol{w}_h) &= -([p_{h,c}] + [p_{h,d}], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} = -([p_{h,d}], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} \\ &= -([p_{h,d}], \{\bar{\boldsymbol{w}}_h\})_{\mathcal{F}_i} = 0 \end{aligned}$$

and

$$b(p_h, \boldsymbol{z}_h) = (\nabla p_h, \boldsymbol{\pi}_0 \boldsymbol{z}_h)_{\mathcal{T}_h} - ([p_{h,d}], \{\bar{\boldsymbol{z}_h}\})_{\mathcal{F}_i} = ||\tilde{h} \nabla p_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{\frac{1}{2}}[p_h]||^2_{\mathcal{F}_i} = |||p_h|||^2_Q.$$

Now, by applying integration by parts, I obtain as in (3.32)

$$a_1(\boldsymbol{u}_h, \boldsymbol{z}_h) = -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{z}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\bar{\boldsymbol{z}_h}\})_{\mathcal{F}_i} + ([[\bar{\boldsymbol{u}_h}]] \boldsymbol{n}_F, \{\nabla \boldsymbol{z}_h\} \boldsymbol{n}_F)_{\mathscr{F}_h}.$$

By Theorem 2.3.13, I have $\boldsymbol{\pi}_0 \boldsymbol{z}_h = \tilde{h}^2 \nabla p_h, \{\bar{\boldsymbol{z}_h}\} = -\tilde{h}[p_h]$ and $\{\nabla \boldsymbol{z}_h\} \boldsymbol{n}_F = \boldsymbol{0}$. So, by inverse inequality (2.25), trace inequality, Cauchy-Schwarz's and Young's inequalities I obtain

$$\begin{aligned}
a_1(\boldsymbol{u}_h, \boldsymbol{z}_h) &= -(\Delta \boldsymbol{u}_h, \tilde{h}^2 \nabla p_h)_{\mathcal{T}_h} - ([[\nabla \boldsymbol{u}_h]], \tilde{h}[p_h])_{\mathcal{F}_i}, \\
|a_1(\boldsymbol{u}_h, \boldsymbol{z}_h)| &\leq ||\Delta \boldsymbol{u}_h||_{\mathcal{T}_h} \, ||\tilde{h}^2 \nabla p_h||_{\mathcal{T}_h} + ||[[\nabla \boldsymbol{u}_h]]||_{\mathcal{F}_i} \, ||\tilde{h}[p_h]||_{\mathcal{F}_i} \\
&\leq C_I ||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h} \, ||\tilde{h} \nabla p_h||_{\mathcal{T}_h} + M C_T ||\tilde{h}^{-\frac{1}{2}} \nabla \boldsymbol{u}_h||_{\mathcal{F}_i} \, ||\tilde{h}[p_h]||_{\mathcal{F}_i} \\
&\leq M ||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h} (||\tilde{h} \nabla p_h||_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[p_h]||_{\mathcal{F}_i}).
\end{aligned}$$

So, I have

$$a_1(\boldsymbol{u}_h, \boldsymbol{z}_h) \geq -C_2 ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - \frac{1}{4} |||p_h|||^2_Q \tag{3.33}$$

where C_2 is a constant independent of h. In summary I obtained,

$$a_1(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h}, \tag{3.34}$$

$$a_1(\boldsymbol{u}_h, \boldsymbol{w}_h) = ||\widetilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||^2_{\mathscr{F}_h}, \tag{3.35}$$

$$b(p_h, \boldsymbol{w}_h) = 0, \tag{3.36}$$

$$b(p_h, \boldsymbol{z}_h) = |||p_h|||^2_Q, \tag{3.37}$$

$$a_1(\boldsymbol{u}_h, \boldsymbol{z}_h) \geq -C_2||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - \frac{1}{4}|||p_h|||^2_Q. \tag{3.38}$$

where C_2 is a constant independent of h. To prove (3.29) I choose $\boldsymbol{v}_h = A\boldsymbol{u}_h + B\boldsymbol{w}_h + D\boldsymbol{z}_h$ and $q_h = p_h$ with constants $A, B,$ and D to be defined. Then,

$$\begin{aligned}
Y &\equiv a_1(\boldsymbol{u}_h, A\boldsymbol{u}_h + B\boldsymbol{w}_h + D\boldsymbol{z}_h) + b(p_h, A\boldsymbol{u}_h + B\boldsymbol{w}_h + D\boldsymbol{z}_h) \\
&- b(q_h, \boldsymbol{u}_h) + c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, A\boldsymbol{u}_h + B\boldsymbol{w}_h + D\boldsymbol{z}_h) \\
&= Aa_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + Ba_1(\boldsymbol{u}_h, \boldsymbol{w}_h) + Da_1(\boldsymbol{u}_h, \boldsymbol{z}_h) \\
&+ Ab(p_h, \boldsymbol{u}_h) + Bb(p_h, \boldsymbol{w}_h) + Db(p_h, \boldsymbol{z}_h) \\
&- b(p_h, \boldsymbol{u}_h) + Ac(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{u}_h) + Bc(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{w}_h) \\
&+ Dc(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{z}_h).
\end{aligned}$$

Taking $\boldsymbol{v}_h = \boldsymbol{0}$ and $q_h = p_h$ in (3.12) yields

$$b(p_h, \boldsymbol{u}_h) = 0. \tag{3.39}$$

Using (3.8), (3.39) and (3.36) and assuming $A \geq 0$, I obtain

$$\begin{aligned} Y \geq\ & Aa_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + Ba_1(\boldsymbol{u}_h, \boldsymbol{w}_h) + Da_1(\boldsymbol{u}_h, \boldsymbol{z}_h) \\ & + Db(p_h, \boldsymbol{z}_h) + Bc(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{w}_h) + Dc(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{z}_h). \end{aligned} \tag{3.40}$$

Now plugging (3.24), (3.25), (3.34), (3.35), (3.37), and (3.38) in (3.40) I obtain

$$\begin{aligned} Y \geq & ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h}(A - DC_2 - BN_0|||\widetilde{\boldsymbol{u}}_h||| - \frac{DN_1}{2}|||\widetilde{\boldsymbol{u}}_h|||^2) \\ & + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}(B(1 - N_0|||\widetilde{\boldsymbol{u}}_h|||) - \frac{DN_1}{2}|||\widetilde{\boldsymbol{u}}_h|||^2) \\ & + |||p_h|||^2_Q(\frac{D}{4}). \end{aligned}$$

I choose the following

$$\begin{aligned} D &= 2, \\ B &= \frac{1 + N_1|||\widetilde{\boldsymbol{u}}_h|||^2}{1 - N_0|||\widetilde{\boldsymbol{u}}_h|||}, \\ A &= 2C_2 + 1 + N_1|||\widetilde{\boldsymbol{u}}_h|||^2 + BN_0|||\widetilde{\boldsymbol{u}}_h|||. \end{aligned}$$

I observe that this implies

$$Y \geq \frac{1}{2}(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h} + |||p_h|||^2_Q)$$

or with (2.42)

$$Y \geq \frac{1}{2}\min(C_E, 1)(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h} + |||p_h|||^2_Q).$$

I conclude that the constant M_3 is a constant independent of h and of $\widetilde{\boldsymbol{u}}_h$. It remains to check that $A > 0$. Since the function $\widetilde{\boldsymbol{u}}_h$ satisfies (3.28), I see that

$$1 \leq \frac{1}{1 - N_0|||\widetilde{\boldsymbol{u}}_h|||} \leq 2$$

So

$$1 + N_1|||\widetilde{\boldsymbol{u}}_h|||^2 \leq B \leq 2(1 + N_1|||\widetilde{\boldsymbol{u}}_h|||^2). \tag{3.41}$$

This implies that $B \geq 0$ and therefore $A \geq 0$. I now prove (3.30).

$$|||\boldsymbol{v}_h||| \leq A|||\boldsymbol{u}_h||| + B|||\boldsymbol{w}_h||| + 2|||\boldsymbol{z}_h|||.$$

Using (3.26) and (3.27) I have

$$|||\boldsymbol{v}_h||| \leq (A + BM)|||\boldsymbol{u}_h||| + 2M|||p_h|||_Q.$$

Using (3.28) with (3.41) I see that

$$0 \leq B \leq 2(1 + \frac{N_1}{4N_0^2}).$$

This implies that

$$0 \leq A \leq C_2 + 3(1 + \frac{N_1}{4N_0^2}).$$

So the constant M_4 is independent of $\widetilde{\boldsymbol{u}}_h$ and independent of h. Since the inf-sup constant is $\beta = M_3/M_4$, I proved that β is a constant independent of h and $\widetilde{\boldsymbol{u}}_h$.

3.3.2 Symmetric Case

I first prove a lemma similar to Lemma 3.3.2

Lemma 3.3.2. *Let $\boldsymbol{w}_h \in \boldsymbol{V}_{bs}$ be the projection defined in Theorem 2.3.13 with the* *arguments*

$$\boldsymbol{a}_h = \mathbf{0}\ ,\ \boldsymbol{b}_h = -\widetilde{h}^{-1}[[\bar{\boldsymbol{u}_h}]]\boldsymbol{n}_F, \quad \textit{and} \quad \boldsymbol{c}_h = \mathbf{0} \tag{3.42}$$

and let $\boldsymbol{z}_h$ be the projection defined in Theorem 2.3.13 with the arguments

$$\boldsymbol{a}_h = \tilde{h}^2 \nabla p_h,\ \boldsymbol{b}_h = \mathbf{0} \textit{ and } \boldsymbol{c}_h = -\tilde{h}[p_h] \tag{3.43}$$

then,

$$c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{w}_h) \geq -N_0 |||\widetilde{\boldsymbol{u}_h}|||(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}) \tag{3.44}$$

and

$$c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{z}_h) \geq -\frac{N_1}{2} |||\widetilde{\boldsymbol{u}_h}|||^2(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}) - \frac{1}{2}|||p_h|||^2_Q. \tag{3.45}$$

where N_0 and N_1 are constants independent of h. In fact, $N_0 = M_P C_0 / C_E$ where C_E is the constant in (2.42) and C_0 is the constant in (3.7).

The proof is almost identical to the proof of Lemma 3.3.1. Assume now that $\widetilde{\boldsymbol{u}}_h \in \boldsymbol{V}_{bs}$ satisfies (3.18). Next, I break the proof of Proposition 3.2.3 in the following two parts: there exist constants M_3 and M_4 independent of h and $\widetilde{\boldsymbol{u}}_h$ such that for each fixed pair $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_{bs} \times Q_h$ there exists a pair $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h$ with

$$M_3 |||(\boldsymbol{u}_h, p_h)|||^2 \leq a_{-1}(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) - b(q_h, \boldsymbol{u}_h) + c(\widetilde{\boldsymbol{u}_h}, \widetilde{\boldsymbol{u}_h}, \boldsymbol{u}_h, \boldsymbol{v}_h) \tag{3.46}$$

and

$$|||(\boldsymbol{v}_h, q_h)||| \leq M_4 |||(\boldsymbol{u}_h, p_h)|||. \tag{3.47}$$

The proof of (3.46) and (3.47) is similar to the proof of (3.29) and (3.30). Let $\boldsymbol{w}_h, \boldsymbol{z}_h$

be defined as Lemma 3.3.2. I fix $\boldsymbol{u}_h$ in $\boldsymbol{V}_{bs}$ and p_h in Q_h.

$$a_{-1}(\boldsymbol{u}_h, \boldsymbol{u}_h) = ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - 2(\{\nabla \boldsymbol{u}_h\}, [[\boldsymbol{u}_h]])_{\mathscr{F}_h}. \tag{3.48}$$

Also, from (2.19), Corollary 2.3.5 and definition of average (2.30)

$$(\{\nabla \boldsymbol{u}_h\}, [[\boldsymbol{u}_h]])_{\mathscr{F}_h} = (\{\nabla \boldsymbol{u}_h\}\boldsymbol{n}_F, [[\boldsymbol{u}_h]]\boldsymbol{n}_F)_{\mathscr{F}_h} = (\{\nabla \boldsymbol{u}_h\}, [[\bar{\boldsymbol{u}}_h]])_{\mathscr{F}_h}.$$

Putting these together and using the Cauchy-Schwarz's inequality, the trace inequality (2.27) and Young's inequality I obtain

$$\begin{aligned}
a_{-1}(\boldsymbol{u}_h, \boldsymbol{u}_h) &\geq ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - 2||\tilde{h}^{\frac{1}{2}}\{\nabla \boldsymbol{u}_h\}||_{\mathscr{F}_h}||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||_{\mathscr{F}_h} \\
&\geq ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - 2C_T||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h}||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||_{\mathscr{F}_h} \\
&\geq \frac{1}{2}||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} - 2C_T^2||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||^2_{\mathscr{F}_h}.
\end{aligned}$$

Using (2.24), (2.19), Corollary 2.3.5, Lemma 2.3.3 and $\boldsymbol{L}^2$ projection $\boldsymbol{\pi}_0$ on $\boldsymbol{V}_h^0$, (see (2.32)) I have,

$$\begin{aligned}
a_{-1}(\boldsymbol{u}_h, \boldsymbol{w}_h) &= -(\Delta \boldsymbol{u}_h, \boldsymbol{w}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\boldsymbol{w}_h\})_{\mathcal{F}_i} - ([[\boldsymbol{u}_h]]\boldsymbol{n}_F, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F)_{\mathscr{F}_h} \\
&= -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0\boldsymbol{w}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\bar{\boldsymbol{w}}_h\})_{\mathcal{F}_i} - ([[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F, \{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F)_{\mathscr{F}_h}.
\end{aligned} \tag{3.49}$$

From Theorem 2.3.13, I have: $\boldsymbol{\pi}_0\boldsymbol{w}_h = \boldsymbol{0}$, $\{\bar{\boldsymbol{w}}_h\}|_F = \boldsymbol{0}$ for all $F \in \mathcal{F}_i$ and $\{\nabla \boldsymbol{w}_h\}\boldsymbol{n}_F = -\tilde{h}^{-1}[[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F$ for all $F \in \mathscr{F}_h$. Thus, by (2.38)

$$a_{-1}(\boldsymbol{u}_h, \boldsymbol{w}_h) = ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F||^2_{\mathscr{F}_h} = ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||^2_{\mathscr{F}_h}.$$

Moreover, as in the proof of the inf-sup condition for the non-symmetric case, I have

$$b(p_h, \boldsymbol{w}_h) = 0, \quad b(p_h, \boldsymbol{z}_h) = |||p_h|||_Q^2.$$

Now, by applying integration by parts, I obtain as in (3.49)

$$a_{-1}(\boldsymbol{u}_h, \boldsymbol{z}_h) = -(\Delta \boldsymbol{u}_h, \boldsymbol{\pi}_0 \boldsymbol{z}_h)_{\mathcal{T}_h} + ([[\nabla \boldsymbol{u}_h]], \{\bar{\boldsymbol{z}}_h\})_{\mathcal{F}_i} - ([[\bar{\boldsymbol{u}}_h]]\boldsymbol{n}_F, \{\nabla \boldsymbol{z}_h\}\boldsymbol{n}_F)_{\mathscr{F}_h}.$$

By Theorem 2.3.13, I have $\boldsymbol{\pi}_0 \boldsymbol{z}_h = \tilde{h}^2 \nabla p_h, \{\bar{\boldsymbol{z}}_h\} = -\tilde{h}[p_h]$ and $\{\nabla \boldsymbol{z}_h\}\boldsymbol{n}_F = \boldsymbol{0}$. So, I obtain the same bound as (3.33),

$$a_{-1}(\boldsymbol{u}_h, \boldsymbol{z}_h) \geq -C_2 ||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h}^2 - \frac{1}{4}|||p_h|||_Q^2. \tag{3.50}$$

where C_2 is a constant independent of h. In summary I obtained,

$$a_{-1}(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq \frac{1}{2}||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h}^2 - 2C_T^2||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||_{\mathscr{F}_h}^2, \tag{3.51}$$

$$a_{-1}(\boldsymbol{u}_h, \boldsymbol{w}_h) = ||\widetilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}}_h]]||_{\mathscr{F}_h}^2, \tag{3.52}$$

$$b(p_h, \boldsymbol{w}_h) = 0, \tag{3.53}$$

$$b(p_h, \boldsymbol{z}_h) = |||p_h|||_Q^2, \tag{3.54}$$

$$a_{-1}(\boldsymbol{u}_h, \boldsymbol{z}_h) \geq -C_2||\nabla \boldsymbol{u}_h||_{\mathcal{T}_h}^2 - \frac{1}{4}|||p_h|||_Q^2. \tag{3.55}$$

where C_2 is a constant independent of h. To prove (3.29) I choose $\boldsymbol{v}_h = A\boldsymbol{u}_h + B\boldsymbol{w}_h + D\boldsymbol{z}_h$ and $q_h = p_h$ with constants $A, B,$ and D to be defined. Then, I obtain the same bound as (3.40) assuming that $A \geq 0$. Now plugging (3.44), (3.45), (3.51), (3.52), (3.54), and (3.55) in (3.40) I obtain

$$
\begin{aligned}
Y \geq & ||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h}(\frac{1}{2}A - C_2 D - BN_0|||\widetilde{\boldsymbol{u}}_h||| - \frac{DN_1}{2}|||\widetilde{\boldsymbol{u}}_h|||^2) \\
& + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h}(-2AC_T^2 + B(1 - N_0|||\widetilde{\boldsymbol{u}}_h|||) - \frac{DN_1}{2}|||\widetilde{\boldsymbol{u}}_h|||^2) \\
& + |||p_h|||^2_Q(\frac{D}{4}).
\end{aligned}
$$

I choose the following

$$
\begin{aligned}
D &= 2, \\
B &= \frac{1 + N_1|||\widetilde{\boldsymbol{u}}_h|||^2(1 + 4C_T^2) + 4C_T^2(1 + 2C_2)}{1 - (1 + 4C_T^2)N_0|||\widetilde{\boldsymbol{u}}_h|||}, \\
A &= 2 + 4C_2 + 2N_1|||\widetilde{\boldsymbol{u}}_h|||^2 + 2BN_0|||\widetilde{\boldsymbol{u}}_h|||.
\end{aligned}
$$

I observe that this implies

$$
Y \geq (||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h} + |||p_h|||^2_Q)
$$

or with (2.42)

$$
Y \geq \min(C_E, 1)(||\nabla \boldsymbol{u}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-\frac{1}{2}}[[\bar{\boldsymbol{u}_h}]]||^2_{\mathscr{F}_h} + |||p_h|||^2_Q)
$$

I conclude that the constant M_3 is a constant independent of h and of $\widetilde{\boldsymbol{u}}_h$. It remains to check that $A > 0$. Since the function $\widetilde{\boldsymbol{u}}_h$ satisfies (3.18), I see that

$$
1 \leq \frac{1}{1 - N_0(1 + 4C_T^2)|||\widetilde{\boldsymbol{u}}_h|||} \leq 2.
$$

So

$$1+N_1(1+4C_T^2)|||\widetilde{\boldsymbol{u}}_h|||^2+4C_T^2(1+2C_2) \leq B \leq 2(1+N_1(1+4C_T^2)|||\widetilde{\boldsymbol{u}}_h|||^2+4C_T^2(1+2C_2)). \tag{3.56}$$

This implies that $B \geq 0$ and therefore $A \geq 0$. I now prove (3.47).

$$|||\boldsymbol{v}_h||| \leq A|||\boldsymbol{u}_h||| + B|||\boldsymbol{w}_h||| + 2|||\boldsymbol{z}_h|||.$$

Using (3.26) and (3.27) I have

$$|||\boldsymbol{v}_h||| \leq (A + BM)|||\boldsymbol{u}_h||| + 2M|||p_h|||_Q.$$

Using (3.18) with (3.56) I see that

$$0 \leq B \leq 2 + 8C_T^2(1 + 2C_2) + \frac{N_1}{2N_0^2(1 + 4C_T^2)})$$

and

$$0 \leq A \leq 2 + 4C_2 + \frac{N_1}{2N_0^2(1 + 4C_T^2)^2} + \frac{B}{(1 + 4C_T^2)}.$$

So the constant M_4 is independent of $\widetilde{\boldsymbol{u}}_h$ and independent of h. Since the inf-sup constant is $\beta = M_3/M_4$, I proved that β is a constant independent of h and $\widetilde{\boldsymbol{u}}_h$.

3.4 Proof of Continuity of G

In this section I prove that G is continuous. Let $(\widetilde{\boldsymbol{u}_n})_{n\geq 0}$ be a sequence in $\boldsymbol{X}_h$ that converges to $\tilde{\boldsymbol{u}}$ in $\boldsymbol{X}_h$. Let $(\widetilde{p_n})_{n\geq 0}$ be a sequence in Q_h that converges to $\tilde{p}$ in Q_h. By definition set $(\boldsymbol{u}_n, p_n) = G(\widetilde{\boldsymbol{u}_n}, \widetilde{p_n})$ and $(\boldsymbol{u}, p) = G(\widetilde{\boldsymbol{u}}, \widetilde{p})$. I will show that the sequence $(\boldsymbol{u}_n)_{n\geq 0}$ converges to $\boldsymbol{u}$ in $\boldsymbol{X}_h$ and the sequence $(p_n)_{n\geq 0}$ converges to p in

Q_h.

By definition I have

$$a_\epsilon(\boldsymbol{u}_n, \boldsymbol{v}) + b(p_n, \boldsymbol{v}) + c(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}_n}, \boldsymbol{u}_n, \boldsymbol{v}) - b(q, \boldsymbol{u}_n) = F(\boldsymbol{v}, q), \tag{3.57}$$

$$a_\epsilon(\boldsymbol{u}, \boldsymbol{v}) + b(p, \boldsymbol{v}) + c(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - b(q, \boldsymbol{u}) = F(\boldsymbol{v}, q). \tag{3.58}$$

By subtracting (3.57) from (3.58) I obtain,

$$a_\epsilon(\boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{v}) + b(p - p_n, \boldsymbol{v}) - b(q, \boldsymbol{u} - \boldsymbol{u}_n) + c(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - c(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}_n}, \boldsymbol{u}, \boldsymbol{v}) = 0. \tag{3.59}$$

I decompose the form c into a trilinear part and a nonlinear part. I write

$$c(\boldsymbol{z}_h, \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h) = d(\boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h) + \ell(\boldsymbol{z}_h, \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h)$$

where the form ℓ is

$$\ell(\boldsymbol{z}_h, \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h) = \sum_{\tau \in \mathcal{T}_h} \int_{\partial \tau_-^{\boldsymbol{z}_h}} |\{\boldsymbol{u}_h\} \cdot \boldsymbol{n}_\tau| (\boldsymbol{v}_h^{int} - \boldsymbol{v}_h^{ext}) \cdot \boldsymbol{w}_h^{int}.$$

I note that

$$d(\widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - d(\widetilde{\boldsymbol{u}_n}, \boldsymbol{u}_n, \boldsymbol{v}) = d(\widetilde{\boldsymbol{u}_n}, \boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{v}) + d(\widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}_n}, \boldsymbol{u}, \boldsymbol{v}) \tag{3.60}$$

and

$$\begin{aligned} \ell(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - \ell(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}_n}, \boldsymbol{u}_n, \boldsymbol{v}) &= \ell(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}} - \widetilde{\boldsymbol{u}_n}, \boldsymbol{u}, \boldsymbol{v}) + \ell(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}_n}, \boldsymbol{u} - \boldsymbol{u}_n, \boldsymbol{v}) \\ &\quad + \ell(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - \ell(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}). \end{aligned} \tag{3.61}$$

Using (3.60) and (3.61) in (3.59), I have

$$a_\epsilon(\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{v})+b(p-p_n,\boldsymbol{v})-b(q,\boldsymbol{u}-\boldsymbol{u}_n)+d(\widetilde{\boldsymbol{u}_n},\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{v})+\ell(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}_n},\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{v})$$
$$=-d(\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}},\boldsymbol{v})-\ell(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}_n},\boldsymbol{u},\boldsymbol{v})-\ell(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v})+\ell(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v}).$$

From the inf-sup condition (3.19), I have

$$\begin{aligned}
&\beta|||(\boldsymbol{u}-\boldsymbol{u}_n,p-p_n)||| \leq \\
&\sup_{(\boldsymbol{v},q)\in\boldsymbol{V}_{bs}\times Q_h}\frac{a_\epsilon(\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{v})+b(p-p_n,\boldsymbol{v})-b(q,\boldsymbol{u}-\boldsymbol{u}_n)+c(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}_n},\boldsymbol{u}-\boldsymbol{u}_n,\boldsymbol{v})}{|||(\boldsymbol{v},q)|||}\\
=&\sup_{(\boldsymbol{v},q)\in\boldsymbol{V}_{bs}\times Q_h}\frac{-d(\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}},\boldsymbol{v})-\ell(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}_n},\boldsymbol{u},\boldsymbol{v})-\ell(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v})+\ell(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v})}{|||(\boldsymbol{v},q)|||}.
\end{aligned} \tag{3.62}$$

Using the same arguments as in Proposition 4.1 of [11], I can show there is a constant M independent of h such that

$$|d(\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}},\boldsymbol{v})|\leq M|||\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}}_n|||\,|||\widetilde{\boldsymbol{u}}|||\,|||\boldsymbol{v}|||,$$
$$|\ell(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}_n},\boldsymbol{u},\boldsymbol{v})|\leq M|||\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}}_n|||\,|||\boldsymbol{u}|||\,|||\boldsymbol{v}|||.$$

Therefore, (3.62) becomes

$$\begin{aligned}
\beta|||(\boldsymbol{u}-\boldsymbol{u}_n,p-p_n)||| \leq M(|||\widetilde{\boldsymbol{u}}|||+|||\boldsymbol{u}|||)|||\widetilde{\boldsymbol{u}}-\widetilde{\boldsymbol{u}}_n|||&\\
+\sup_{\boldsymbol{v}\in\boldsymbol{V}_{bs}}\frac{\ell(\widetilde{\boldsymbol{u}_n},\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v})-\ell(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v})}{|||\boldsymbol{v}|||}&.
\end{aligned}$$

The inf-sup constant β is independent of $\widetilde{\boldsymbol{u}}_h$. To conclude that G is continuous, it suffices to show that

$$\lim_{n\to\infty} \sup_{\boldsymbol{v}\in \boldsymbol{V}_{bs}} \frac{\ell(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - \ell(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v})}{|||\boldsymbol{v}|||} = 0.$$

This is proved in the following lemma.

Lemma 3.4.1. *Assume that* $(\widetilde{\boldsymbol{u}}_n)_n$ *converges to* $\widetilde{\boldsymbol{u}}$ *in* $\boldsymbol{V}_{bs}$. *Then for any* $\boldsymbol{u} \in \boldsymbol{V}_{bs}$, *I have*

$$\lim_{n\to\infty} \sup_{\boldsymbol{v}\in \boldsymbol{V}_{bs}} \frac{\ell(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - \ell(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v})}{|||\boldsymbol{v}|||} = 0. \tag{3.63}$$

Proof. Denote by θ the integrand of $\ell(\widetilde{\boldsymbol{u}_n}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v})$. I have

$$\ell(\widetilde{\boldsymbol{u}}_n, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) - \ell(\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}}, \boldsymbol{u}, \boldsymbol{v}) = \sum_{\tau\in\mathcal{T}_h} \Big(\int_{\partial\tau_-^{\widetilde{\mathbf{u}}_n}} \theta - \int_{\partial\tau_-^{\widetilde{\mathbf{u}}}} \theta \Big).$$

I now fix an element $\tau \in \mathcal{T}_h$ and consider one face $F \in \partial\tau$. I recall the definition (3.6) (choosing $\boldsymbol{n}_F = \boldsymbol{n}_\tau$)

$$\partial\tau_-^{\widetilde{\mathbf{u}}} = \{\boldsymbol{x} \in \partial\tau \ : \ \{\widetilde{\boldsymbol{u}}\} \cdot \boldsymbol{n}_F < 0\},$$
$$\partial\tau_-^{\widetilde{\mathbf{u}}_n} = \{\boldsymbol{x} \in \partial\tau \ : \ \{\widetilde{\boldsymbol{u}}_n\} \cdot \boldsymbol{n}_F < 0\}.$$

From the definition of $\boldsymbol{V}_{bs}$ the functions $\{\widetilde{\boldsymbol{u}}\}\cdot\boldsymbol{n}_F$ and $\{\widetilde{\boldsymbol{u}}_n\cdot\boldsymbol{n}_F\}$ are quadratic functions on F. Define

$$W = \int_{F\cap\partial\tau_-^{\widetilde{\mathbf{u}}_n}} \theta - \int_{F\cap\partial\tau_-^{\widetilde{\mathbf{u}}}} \theta$$

From equivalence of norms I have

$$\lim_{n\to\infty} \|\{\widetilde{\boldsymbol{u}}\} \cdot \boldsymbol{n}_F - \{\widetilde{\boldsymbol{u}}_n\} \cdot \boldsymbol{n}_F\|_{L^\infty(F)} = 0. \tag{3.64}$$

Clearly this implies that if the quadratic function $\{\widetilde{\boldsymbol{u}}\}\cdot\boldsymbol{n}_F$ never vanishes on $\bar{F}$, then there exists $n_0 \geq 0$ so that for all $n \geq n_0$ the function $\{\widetilde{\boldsymbol{u}}_n\}\cdot\boldsymbol{n}_F$ has the same strict sign than $\{\widetilde{\boldsymbol{u}}\}\cdot\boldsymbol{n}_F$. This means that $W = 0$. In the other case where the quadratic function $\{\widetilde{\boldsymbol{u}}\}\cdot\boldsymbol{n}_F$ vanishes to at least one point on $\bar{F}$, I write using the characteristic function

$$W = \int_F \theta(\chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}_n}} - \chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}}}).$$

This gives by expanding the definition of θ

$$W \leq \|\{\widetilde{\boldsymbol{u}}\cdot\boldsymbol{n}_\tau\}\|_{L^4(F)}\|j(\boldsymbol{u})\|_{L^4(F)}\|\boldsymbol{v}^{\mathrm{int}}\|_{L^4(F)}\|\chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}_n}} - \chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}}}\|_{L^4(F)}.$$

Using trace inequalities, I have for constants M_F^* independent of $\boldsymbol{v}$ and n

$$\sum_{\tau\in\mathcal{T}_h}\sum_{F\in\partial\tau}\int_F|\theta||\chi_{S_F^n}|_F \leq |||\boldsymbol{v}|||\sum_{\tau\in\mathcal{T}_h}\sum_{F\in\partial\tau} M_F^*\|\chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}_n}} - \chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}}}\|_{L^4(F)}.$$

This implies that

$$\sup_{\boldsymbol{v}\in\boldsymbol{V}_{bs}}\frac{\ell(\widetilde{\boldsymbol{u}}_n,\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v}) - \ell(\widetilde{\boldsymbol{u}},\widetilde{\boldsymbol{u}},\boldsymbol{u},\boldsymbol{v})}{|||\boldsymbol{v}|||} \leq \sum_{\tau\in\mathcal{T}_h}\sum_{F\in\partial\tau} M_F^*\|\chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}_n}} - \chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}}}\|_{L^4(F)}.$$

It suffices then to show that

$$\lim_{n\to\infty}\|\chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}_n}} - \chi_{F\cap\partial\tau_-^{\tilde{\mathbf{u}}}}\|_{L^4(F)} = 0.$$

The quadratic function $\{\widetilde{\boldsymbol{u}}\}\cdot\boldsymbol{n}_F$ vanishes at one or two points located in $\bar{F}$. Fix $\epsilon > 0$. From (3.64), I see that there exists $n_0 \geq 0$ so that for all $n \geq n_0$ the function $\{\widetilde{\boldsymbol{u}}_n\}\cdot\boldsymbol{n}_F$ has the same strict sign than $\{\widetilde{\boldsymbol{u}}\}\cdot\boldsymbol{n}_F$ on F except on one or two (at most) intervals whose total length is bounded by ϵ. So I have

$$\|\chi_{F\cap\partial_T \underline{\tilde{u}}_n} - \chi_{F\cap\partial_T \underline{\tilde{u}}}\|^4_{L^4(F)} \leq \epsilon$$

which concludes the proof. □

3.5 Error Estimates

In this section, error estimates are derived. We recall an approximation operator $\boldsymbol{R}_h : \boldsymbol{H}^1_0(\Omega) \to \boldsymbol{V}_h$ satisfying:

$$b(q_h, \boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}) = 0, \ \forall \boldsymbol{v} \in \boldsymbol{H}^1_0, \ \forall q_h \in Q_h, \tag{3.65}$$

$$|||\boldsymbol{R}_h(\boldsymbol{v}) - \boldsymbol{v}||| \leq C_R h^{s-1} |\boldsymbol{v}|_{\boldsymbol{H}^s(\Omega)}, \ \forall s \in [1,2], \ \forall \boldsymbol{v} \in \boldsymbol{H}^1_0 \cap \boldsymbol{H}^s(\Omega). \tag{3.66}$$

More details on the construction of the operator $\boldsymbol{R}_h$ can be found in [25, 42]. The following theorem establishes an a priori estimate for the numerical solution.

Theorem 3.5.1. *Assume there is a solution $(\boldsymbol{u}, p)$ to 3.1-3.3 such that $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$, $p \in H^1(\Omega)$ and the following bound holds:*

$$|\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} \leq \frac{\beta}{2\sqrt{2}C_0 C_R}. \tag{3.67}$$

Then, there exists a constant M independent of $h, \boldsymbol{u}$ and p such that

$$|||\boldsymbol{v} - \boldsymbol{v}_h||| \leq Mh(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + ||p||_{H^1(\Omega)}).$$

Proof. We decompose the errors into a numerical error and an approximation error. Denote $\pi p \in Q_h$ an optimal approximation of p; for instance, πp could be an interpolant (for example P_α defined in 2.37)or a L^2 projection. Define

$$\boldsymbol{\xi}_u = \boldsymbol{u}_h - \boldsymbol{R}_h(\boldsymbol{u}),\ \xi_p = p_h - \pi p,\ \boldsymbol{\eta}_u = \boldsymbol{u} - \boldsymbol{R}_h(\boldsymbol{u}),\ \eta_p = p - \pi p.$$

The error equation is:

$$\begin{aligned} a_\epsilon(\boldsymbol{\xi}_u, \boldsymbol{v}_h) + b(\xi_p, \boldsymbol{v}_h) - b(q_h, \boldsymbol{\xi}_u) + c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) \\ = a_\epsilon(\boldsymbol{\eta}_u, \boldsymbol{v}_h) + b(\eta_p, \boldsymbol{v}_h) - b(q_h, \boldsymbol{\eta}_u). \end{aligned}$$

Using the fact that $\boldsymbol{u}$ is in $\boldsymbol{H}_0^1(\Omega)$, I write:

$$\begin{aligned} c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - c(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) = c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}_h) - c(\boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}_h) \\ = c(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{\xi}_u, \boldsymbol{v}_h) + c(\boldsymbol{u}_h, \boldsymbol{\xi}_u, \boldsymbol{R}_h(\boldsymbol{u}), \boldsymbol{v}_h) - c(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{R}_h(\boldsymbol{u}), \boldsymbol{v}_h) - c(\boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{\eta}_u, \boldsymbol{v}_h). \end{aligned}$$

Using the inf-sup condition 3.2.3, I obtain:

$$\begin{aligned} \beta|||(\boldsymbol{\xi}_u, \xi_p)||| \le \sup_{(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_{bs} \times Q_h} \frac{1}{|||(\boldsymbol{v}_h, q_h)|||} (a_\epsilon(\boldsymbol{\eta}_u, \boldsymbol{v}_h) + b(\eta_p, \boldsymbol{v}_h) - b(q_h, \boldsymbol{\eta}_u) \\ - c(\boldsymbol{u}_h, \boldsymbol{\xi}_u, \boldsymbol{R}_h(\boldsymbol{u}), \boldsymbol{v}_h) + c(\boldsymbol{u}_h, \boldsymbol{\xi}_u, \boldsymbol{R}_h(\boldsymbol{u}), \boldsymbol{v}_h) + c(\boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{\eta}_u, \boldsymbol{v}_h)). \end{aligned}$$

The term $b(q_h, \boldsymbol{\eta}_u)$ vanishes due to (3.65). From (3.66), I have

$$|||\boldsymbol{R}_h(\boldsymbol{u})||| \le C_R |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)}.$$

With the continuity of c, I obtain

$$c(\boldsymbol{u}_h, \boldsymbol{\xi}_u, \boldsymbol{R}_h(\boldsymbol{u}), \boldsymbol{v}_h) \le C_0 C_R |||\boldsymbol{\xi}_u|||\ |\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)} |||\boldsymbol{v}_h|||.$$

Using arguments similar to those found in [26], I obtain the following bound:

$$a_\epsilon(\boldsymbol{\eta}_u, \boldsymbol{v}_h) + b(\eta_p, \boldsymbol{v}_h) + c(\boldsymbol{u}_h, \boldsymbol{\eta}_u, \boldsymbol{R}_h(\boldsymbol{u}), \boldsymbol{v}_h) + c(\boldsymbol{u}_h, \boldsymbol{u}, \boldsymbol{\eta}_u, \boldsymbol{v}_h)$$
$$\leq Mh(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)})|||\boldsymbol{v}_h|||,$$

where M is a constant independent of $h, \boldsymbol{u}$ and p. Combining the bounds above yields

$$\beta|||(\boldsymbol{\xi}_u \xi_p)||| \leq Mh(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)}) + C_0 C_R |||\boldsymbol{\xi}_u||||\boldsymbol{u}|_{\boldsymbol{H}^1(\Omega)}.$$

Using assumption 3.67, I conclude:

$$\frac{\beta}{2\sqrt{2}}|||\boldsymbol{\xi}_u||| + \beta|||\xi_p|||_Q \leq Mh(|\boldsymbol{u}|_{\boldsymbol{H}^2(\Omega)} + |p|_{H^1(\Omega)}).$$

The final result is obtained by using a triangle inequality and the optimal approximations of $\boldsymbol{R}_h(\boldsymbol{u})$ and πp. □

3.6 Numerical Results

In this section I consider three examples to verify the error estimation results. I considered linears for velocity and piecewise constants for pressure in the following examples. The domain Ω is the unit square.

3.6.1 Example1: Exact Solution in Discrete Space

In this example I consider the exact solution to be in space. The exact solution is given as

$$\boldsymbol{u}(x, y) = \begin{bmatrix} \text{-y} \\ \text{x} \end{bmatrix}, p = 0.$$

Therefore, the right-hand side is as following:

$$\boldsymbol{f}(x,y) = \begin{bmatrix} \text{-x} \\ \text{-y} \end{bmatrix}$$

The following tables include the results for non-symmetric and symmetric cases respectively

- Non-symmetric case

Table 3.1 : Non-symmetric DG solution in discrete space

N	$\|\|p - p_h\|\|_{L^2}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_{\boldsymbol{L}^2}$	$\|\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|\|_{\boldsymbol{L}^2}$
2	3.6519081e-12	5.1488477e-13	3.3319588e-12
4	8.3174823e-13	1.0353555e-13	1.2857692e-12
8	2.7740368e-12	6.4228405e-14	1.5600257e-12
16	1.9440667e-11	6.6133715e-14	2.5556108e-12

- Symmetric case

Table 3.2 : Symmetric DG solution in discrete space

N	$\|\|p - p_h\|\|_{L^2}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_{\boldsymbol{L}^2}$	$\|\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|\|_{\boldsymbol{L}^2}$
2	3.0868794e-12	2.4715756e-12	1.8624607e-11
4	4.4992222e-13	5.3487424e-13	7.7643988e-12
8	2.6655567e-13	3.9290368e-14	1.0958917e-12
16	9.4661669e-13	1.1972030e-13	5.7580303e-12

3.6.2 Example 2: Exact Polynomial Solution not in Discrete Space

In this example I consider a polynomial solution which is not in the discrete space. The exact solution is:

$$\boldsymbol{u}(x,y) = \begin{bmatrix} y^2 - 2y + 2x \\ x^2 - x - 2y \end{bmatrix},$$

$$p = 0.$$

Also, the right hand side is

$$\boldsymbol{f}(x,y) = \begin{bmatrix} -2y^2 + 6x + 2x^2y - 2xy - 2x^2 - 2 \\ -y^2 + 2x^2 + 2xy^2 - 4xy + 6y - 2 \end{bmatrix}.$$

The results for non-symmetric and symmetric cases are shown in the following tables respectively

- Non-symmetric case

Table 3.3 : Non-symmetric DG solution not in discrete space

N	$\|\|p - p_h\|\|_{L^2}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_{\boldsymbol{L}^2}$	$\|\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|\|_{\boldsymbol{L}^2}$
2	1.8400453e-01	2.2839403e-02	2.6373823e-01
4	1.0912133e-01 (0.7538)	6.4857633e-03 (1.8162)	1.3785238e-01 (0.9360)
8	6.0242442e-02 (0.8571)	1.7160162e-03 (1.9182)	7.0635726e-02 (0.9647)
16	3.1594532e-02 (0.9311)	4.3788005e-04 (1.9705)	3.5729151e-02 (0.9833)

- Symmetric case

Table 3.4 : Symmetric DG solution not in discrete space

N	$\|\|p - p_h\|\|_{L^2}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_{\boldsymbol{L}^2}$	$\|\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|\|_{\boldsymbol{L}^2}$
2	1.5644276e-01	1.0098142e-01	7.4552877e-01
4	1.0050411e-01 (0.64)	9.4707706e-03 (3.47)	1.5060992e-01 (2.88)
8	6.1967413e-02 (0.69)	2.6928698e-03 (1.81)	7.9355563e-02 (0.93)
16	3.4583606e-02 (0.88)	7.1413905e-04 (1.91)	4.0725964e-02 (0.96)

3.6.3 Example 3: Taylor-Green Vortex

In first example I consider a Taylor-Green vortex solution given by:

$$\boldsymbol{u}(x,y) = \begin{bmatrix} \sin(x)\cos(y) \\ -\cos(x)\sin(y) \end{bmatrix},$$

$$p = 0.$$

Therefore, the right-hand side is:

$$\boldsymbol{f}(x,y) = \begin{bmatrix} \sin(y)\cos(x) + 2\sin(x)\cos(y) \\ \cos(y)\sin(y) - 2\cos(x)\cos(y) \end{bmatrix}$$

The results are shown in the following tables. Table 3.5 presents the results for non-symmetric case and table 3.6 shows results of the symmetric case.

- Non-symmetric case

Table 3.5 : Non-symmetric Taylor-Green vortex

N	$\|\|p - p_h\|\|_{L^2}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_{\boldsymbol{L}^2}$	$\|\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|\|_{\boldsymbol{L}^2}$
2	3.3910872e-02	1.1465840e-02	1.1796583e-01
4	1.4475169e-02 (1.2282)	2.9392630e-03 (1.9638)	5.9912170e-02 (0.9774)
8	6.9344989e-03 (1.0617)	7.4570037e-04 (1.9788)	3.0269120e-02 (0.9850)
16	3.4405274e-03 (1.0112)	1.8785696e-04 (1.9890)	1.5221147e-02 (0.9918)

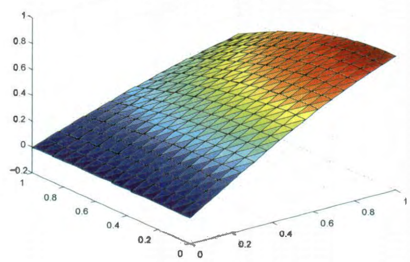
- Symmetric case

Table 3.6 : Symmetric Taylor-Green vortex

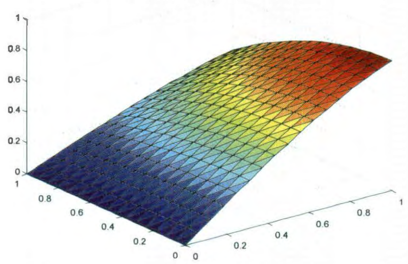
N	$\|\|p - p_h\|\|_{L^2}$	$\|\|\boldsymbol{u} - \boldsymbol{u}_h\|\|_{\boldsymbol{L}^2}$	$\|\|\nabla(\boldsymbol{u} - \boldsymbol{u}_h)\|\|_{\boldsymbol{L}^2}$
2	2.0663622e-02	1.3724891e-02	1.2748168e-01
4	9.2696429e-03(1.1565)	3.2590190e-03 (2.0743)	6.2426091e-02 (1.0301)
8	4.2125703e-03 (1.1378)	8.1832453e-04 (1.9937)	3.1329833e-02 (0.9946)
16	2.0184211e-03 (1.0615)	2.0597041e-04 (1.9902)	1.5732883e-02 (0.9938)

I present the figures for the Taylor-Green vortex case. Respectively, figures 3.1,3.2,3.3 are the first component of velocity, the second component of velocity, and the velocity of the non-symmetric case. In the order given, Figures 3.4,3.5,3.6 present the first component of velocity, the second component of velocity and the velocity of the symmetric case, respectively. Finally, figure 3.7 is an example of the mesh I used.

Figure 3.1 : Non-symmetric: First component of velocity

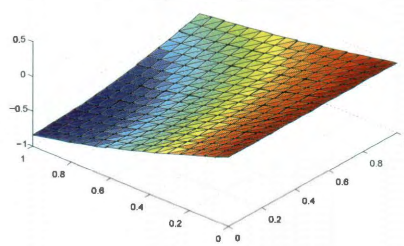


(a) Non-symmetric: The first component of approximate velocity

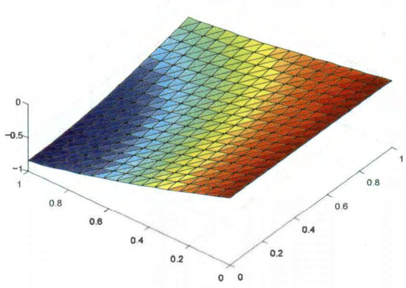


(b) Non-symmetric: The first component of exact velocity

Figure 3.2 : Non-symmetric: Second component of velocity



(a) Non-symmetric: The second component of approximate velocity



(b) Non-symmetric: The second component of exact velocity

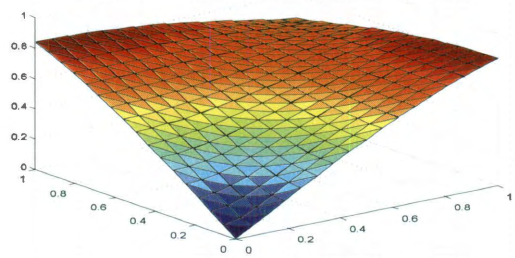
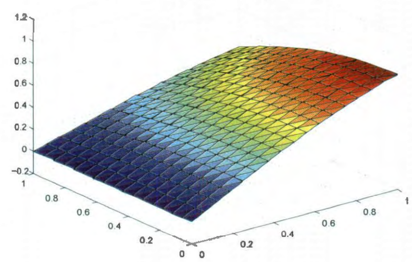
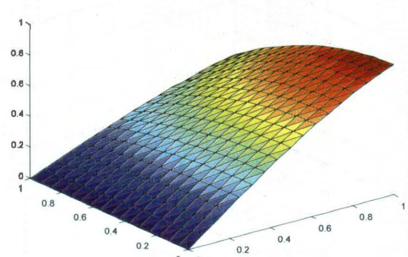


Figure 3.3 : Non-symmetric: Velocity norm

Figure 3.4 : Symmetric: First component of velocity

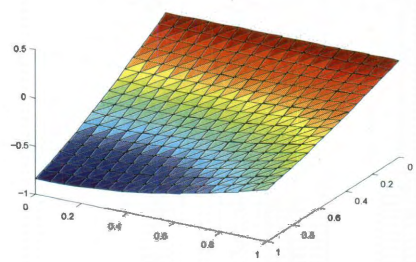


(a) Symmetric: The first component of approximate velocity

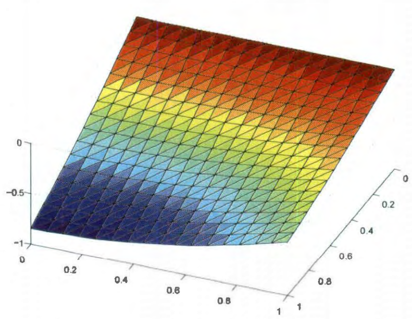


(b) Symmetric: The first component of exact velocity

Figure 3.5 : Symmetric: Second component of velocity



(a) Symmetric: The second component of approximate velocity



(b) Symmetric: The second component of exact velocity

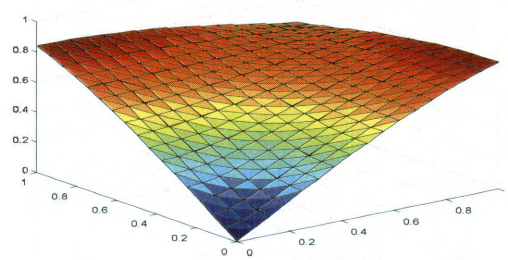


Figure 3.6 : Symmetric: Velocity norm

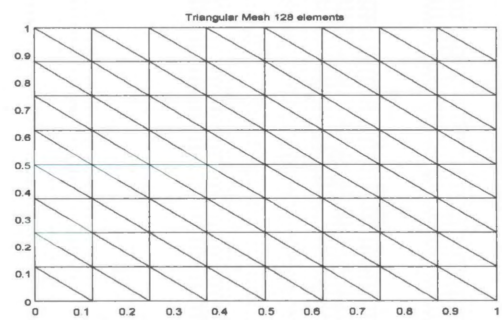


Figure 3.7 : Example of the mesh used in the simulations

Chapter 4

Conclusion

In this work I used low order penalty-free discontinuous Galerkin (DG) methods for numerically solving the steady incompressible Stokes and Navier-Stokes equations. I have established optimal a priori error estimates in broken energy norm for both symmetric and non-symmetric schemes in 2D and 3D.

I first considered non-symmetric penalty-free DG scheme for the Stokes equations, when piecewise polynomials were used to approximate the velocity. To overcome the instability of standard DG we used locally quadratic bubbles to enrich the velocity space. I verified the existence, uniqueness of the solution and proved optimal convergence in the broken energy norm.

Then, I formulated the bubble stabilized symmetric and non-symmetric scheme for steady incompressible Navier-Stokes equations in the special case of piecewise linear approximation. I used the upwind discretization for the non-linear term. The standard DG with penalty equal to zero is unstable in the sense that the linear system in each iteration is non-singular. I obtained a convergent method by enriching the space with locally quadratic polynomials. Since a direct application of the generalized Lax-Milgram theorem is not possible, I restated the solution as a fixed-point of a problem-related map. I proved an optimal error in the broken energy norm for both symmetric and non-symmetric cases. Numerical examples confirmed the theoretical methods.

As future work, I will apply penalty-free discontinuous Galerkin method to solve the coupled problem of free flow with two phase flow in porous media.

Appendix A

Proof of Technical Lemmas

Proof of lemma 2.3.1:

I assume $d = 2$. The proof for $d = 3$ is similar.

Fix a face $F \in \mathscr{F}_h$. Assume first it is an interior face shared by elements τ_1, τ_2. Assume that $\boldsymbol{n}_F = \boldsymbol{n}_1$. Then, we write

$$\boldsymbol{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}, \boldsymbol{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix}, \boldsymbol{n}_1 = \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix}, \boldsymbol{n}_2 = \begin{bmatrix} n_{21} \\ n_{22} \end{bmatrix}$$

Then by definition (2.6), we obtain

$$[[\boldsymbol{v}]] : \nabla \boldsymbol{w} = (\boldsymbol{v}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{n}_2) : \nabla \boldsymbol{w}$$

$$= \left(\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \otimes \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} + \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} \otimes \begin{bmatrix} n_{21} \\ n_{22} \end{bmatrix} \right) : \begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix}$$
$$= \begin{bmatrix} v_{11}n_{11} & v_{11}n_{12} \\ v_{12}n_{11} & v_{12}n_{12} \end{bmatrix} + \begin{bmatrix} v_{21}n_{21} & v_{21}n_{22} \\ v_{22}n_{21} & v_{22}n_{22} \end{bmatrix} : \begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix}$$

$$= (v_{11}n_{11} + v_{21}n_{21})\frac{\partial w_1}{\partial x} + (v_{11}n_{12} + v_{21}n_{22})\frac{\partial w_1}{\partial y} +$$
$$(v_{12}n_{11} + v_{22}n_{21})\frac{\partial w_2}{\partial x} + (v_{12}n_{12} + v_{22}n_{22})\frac{\partial w_2}{\partial y}. \quad \text{(A.1)}$$

On the other hand, we have since $n_2 = -n_1$:

$$[[\boldsymbol{v}]]\boldsymbol{n}_F = \begin{bmatrix} (v_{11}n_{11} + v_{21}n_{21})n_{11} + (v_{11}n_{12} + v_{21}n_{22})n_{12} \\ (v_{12}n_{11} + v_{22}n_{21})n_{11} + (v_{12}n_{12} + v_{22}n_{22})n_{12} \end{bmatrix}$$
$$= \begin{bmatrix} v_{11} - v_{21} \\ v_{12} - v_{22} \end{bmatrix}$$

and

$$[[\boldsymbol{v}]]\boldsymbol{n}_F \cdot \nabla \boldsymbol{w} \boldsymbol{n}_F = (v_{11} - v_{21})(\frac{\partial w_1}{\partial x} n_{11} + \frac{\partial w_1}{\partial y} n_{12}) + (v_{12} - v_{22})(\frac{\partial w_2}{\partial x} n_{11} + \frac{\partial w_2}{\partial y} n_{12}) \quad \text{(A.2)}$$

which is the same expression as (A.1). So we have proved (2.19). Next we prove the other results.

$$[[\boldsymbol{v}]]\boldsymbol{n}_F \otimes \boldsymbol{n}_F = \begin{bmatrix} v_{11} - v_{21} \\ v_{12} - v_{22} \end{bmatrix} \otimes \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix}$$
$$= \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \otimes \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} + \begin{bmatrix} -v_{21} \\ -v_{22} \end{bmatrix} \otimes \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix}$$
$$= \boldsymbol{v}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{n}_2 = [[\boldsymbol{v}]].$$

Finally, we have

$$[[\boldsymbol{v}]]\boldsymbol{n}_1 \cdot \boldsymbol{n}_1 = \begin{bmatrix} v_{11} - v_{21} \\ v_{12} - v_{22} \end{bmatrix} \cdot \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix}$$

$$= v_{11}n_{11} - v_{21}n_{11} + v_{12}n_{12} - v_{22}n_{12}. \tag{A.3}$$

On the other hand

$$[\boldsymbol{v}] = v_1 n_1 + v_2 n_2 = v_{11}n_{11} + v_{12}n_{12} + v_{21}n_{21} + v_{22}n_{22} \tag{A.4}$$

which is the same expression as (A.3). Next, we consider the case of boundary face. Let $F \in \mathscr{F}_e$ be a fixed boundary face.

$$\begin{aligned}
[[\boldsymbol{v}]]\boldsymbol{n} \cdot \nabla \boldsymbol{w}\boldsymbol{n} &= (\boldsymbol{v} \otimes \boldsymbol{n})\boldsymbol{n} \cdot \nabla \boldsymbol{w}\boldsymbol{n} \\
&= \begin{bmatrix} v_{11}n_{11} & v_{11}n_{12} \\ v_{12}n_{11} & v_{12}n_{12} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial w_1}{\partial x} & \frac{\partial w_1}{\partial y} \\ \frac{\partial w_2}{\partial x} & \frac{\partial w_2}{\partial y} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \\
&= \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial w_1}{\partial x} n_{11} + \frac{\partial w_1}{\partial y} n_{12} \\ \frac{\partial w_2}{\partial x} n_{11} + \frac{\partial w_2}{\partial y} n_{12} \end{bmatrix} \\
&= [[\boldsymbol{v}]] : \nabla \boldsymbol{w}
\end{aligned}$$

So, we have proved (2.19). Next

$$
\begin{aligned}
[[\boldsymbol{v}]]\boldsymbol{n} \otimes \boldsymbol{n} &= (\boldsymbol{v} \otimes \boldsymbol{n})\boldsymbol{n} \otimes \boldsymbol{n} \\
&= \begin{bmatrix} v_{11}n_{11} & v_{11}n_{12} \\ v_{12}n_{11} & v_{12}n_{12} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \otimes \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \\
&= \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \otimes \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \\
&= \boldsymbol{v} \otimes \boldsymbol{n} = [[\boldsymbol{v}]]
\end{aligned}
$$

which is the first part of (2.20).

Finally, we have

$$[\boldsymbol{v}] = \boldsymbol{v} \cdot \boldsymbol{n} = v_{11}n_{11} + v_{12}n_{12} \tag{A.5}$$

Also,

$$
\begin{aligned}
[[\boldsymbol{v}]]\boldsymbol{n} \cdot \boldsymbol{n} &= (\boldsymbol{v} \otimes \boldsymbol{n})\boldsymbol{n} \cdot \boldsymbol{n} \\
&= \begin{bmatrix} v_{11}n_{11} & v_{11}n_{12} \\ v_{12}n_{11} & v_{12}n_{12} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \cdot \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \\
&= \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} \cdot \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \\
&= \boldsymbol{v} \cdot \boldsymbol{n}
\end{aligned}
$$

So, we proved second part of (2.20).

Proof of lemma 2.3.2:

I assume $d = 2$. The proof for $d = 3$ is similar.

Fix a face $F \in \mathscr{F}_h$. Assume first it is an interior face shared by elements τ_1, τ_2. Assume that $\boldsymbol{n}_F = \boldsymbol{n}_1$. Then by (A)

$$\begin{aligned}
\| \, [[\boldsymbol{v}]]\boldsymbol{n}_F \, \|_F^2 &= \left\| \begin{bmatrix} v_{11} - v_{21} \\ v_{12} - v_{22} \end{bmatrix} \right\|_F^2 \\
&= \| \, v_{11} - v_{21} \, \|_F^2 + \| \, v_{12} - v_{22} \, \|_F^2
\end{aligned}$$

I also compute

$$\begin{aligned}
\| \, [[\boldsymbol{v}]] \, \|_F^2 &= \| \, \boldsymbol{v}_1 \otimes \boldsymbol{n}_1 + \boldsymbol{v}_2 \otimes \boldsymbol{n}_2 \, \|_F^2 \\
&= \left\| \begin{bmatrix} (v_{11} - v_{21})n_{11} & (v_{11} - v_{21})n_{12} \\ (v_{12} - v_{22})n_{11} & (v_{12} - v_{22})n_{12} \end{bmatrix} \right\|_F^2 \\
&= \| \, (v_{11} - v_{21})n_{11} \, \|_F^2 + \| \, (v_{11} - v_{21})n_{12} \, \|_F^2 + \| \, (v_{12} - v_{22})n_{11} \, \|_F^2 + \| \, (v_{12} - v_{22})n_{12} \, \|_F^2 \\
&= \Big(\| \, v_{11} - v_{21} \, \|_F^2 + \| \, v_{12} - v_{22} \, \|_F^2 \Big) \Big(n_{11}^2 + n_{12}^2 \Big).
\end{aligned}$$

Since $n_{11}^2 + n_{12}^2 = 1$, we showed the first part of (2.21). Finally we have,

$$\begin{aligned}
\| \, [\boldsymbol{v}] \, \|_F^2 &= \| \, \boldsymbol{v}_1 \cdot \boldsymbol{n}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{n}_2 \, \|_F^2 \\
&= \| \, v_{11}n_{11} + v_{12}n_{12} + v_{21}n_{21} + v_{22}n_{22} \, \|_F^2 \\
&= \| \, (v_{11} - v_{21})n_{11} + (v_{12} - v_{22})n_{12} \, \|_F^2 \\
&\leq 2 \max{(n_{11}^2, n_{12}^2)} \Big(\| \, v_{11} - v_{21} \, \|_F^2 + \| \, v_{12} - v_{22} \, \|_F^2 \Big) \\
&\leq 2 \, \| \, [[\boldsymbol{v}]] \, \|_F^2
\end{aligned}$$

therefore,

$$\sum \parallel [\boldsymbol{v}] \parallel_F^2 \leq \sqrt{2} \sum \parallel [[\boldsymbol{v}]] \parallel_F^2$$

which is the second part of (2.21)

Now, we consider a boundary face F:

$$\begin{aligned}
||[[\boldsymbol{v}]]||_F^2 &= ||\boldsymbol{v} \otimes \boldsymbol{n}||_F^2 \\
&= || \begin{bmatrix} v_{11}n_{11} & v_{11}n_{12} \\ v_{21}n_{11} & v_{21}n_{12} \end{bmatrix} ||_F^2 \\
&= ||v_{11}||^2(||n_{11}||^2 + ||n_{12}||^2) + ||v_{21}||^2(||n_{11}||^2 + ||n_{12}||^2) \\
&= ||\boldsymbol{v}||_F^2
\end{aligned}$$

therefore,

$$\parallel [[\boldsymbol{v}]] \parallel_F^2 = \parallel \boldsymbol{v} \parallel_F^2 \tag{A.6}$$

On the other hand,

$$\begin{aligned}
\parallel (\boldsymbol{v} \otimes \boldsymbol{n})\boldsymbol{n} \parallel_F &= \parallel \begin{bmatrix} v_{11}n_{11} & v_{11}n_{12} \\ v_{21}n_{11} & v_{21}n_{12} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} \parallel_F \\
&= \parallel \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} \parallel_F
\end{aligned}$$

which is the same expression as (A.6). For the second part of (2.21)

$$||[\boldsymbol{v}]||_F = ||\boldsymbol{v}\cdot\boldsymbol{n}||_F$$

$$= ||v_{11}n_{11} + v_{12}n_{12}||_F$$

$$\leq 2\max\{n_{11}^2, n_{12}^2\}(||v_{11}||^2 + ||v_{12}||^2)||\boldsymbol{v}||_F^2$$

$$= 2||[[\boldsymbol{v}]]||_F$$

therefore, by summing up over all faces we have proved second part of (2.21).

Proof of lemma 2.3.3:

It is true because if $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}$ then $\boldsymbol{v}_h = \boldsymbol{q} + \boldsymbol{\alpha x}\cdot\boldsymbol{x}$ where $\boldsymbol{q} \in \boldsymbol{V}_h^1$. Since $\Delta(\boldsymbol{\alpha x}.\boldsymbol{x}) = 2d\boldsymbol{\alpha} \in \boldsymbol{V}_h^0$ and $\Delta\boldsymbol{q} = 0$ then $\Delta\boldsymbol{v}_h \in \boldsymbol{V}_h^0$.

Proof of lemma 2.3.4:

Let $\boldsymbol{v}_h \in \boldsymbol{V}_{bs}^1$, and fix one element τ. I can write

$$\boldsymbol{v}_h = \begin{bmatrix} d + bx + cy + \alpha(x^2+y^2) \\ g + ex + fy + \beta(x^2+y^2) \end{bmatrix}$$

Therefore,

$$\nabla v_{h,1} = \begin{bmatrix} b + 2\alpha x \\ c + 2\alpha y \end{bmatrix}$$

and

$$\nabla v_{h,2} = \begin{bmatrix} e + 2\beta x \\ f + 2\beta y \end{bmatrix},$$

which is in the lowest order Raviart-Thomas space.

On the other hand if $\boldsymbol{r}_{h,1} = \begin{bmatrix} a_1 + 2b_1 x \\ a_2 + 2b_2 y \end{bmatrix}$ and $\boldsymbol{r}_{h,2} = \begin{bmatrix} c_1 + 2d_1 x \\ c_2 + 2d_2 y \end{bmatrix}$ then we define $v_{h,1} = a_1 x + b_1 x^2 + a_2 y + b_2 y^2 + m_1, v_{h,2} = c_1 x + d_1 x^2 + c_2 y + d_2 y^2 + m_2$.

Proof of Corollary 2.3.5:

Using (3.4) in [39] since $\nabla \boldsymbol{v}$ is in Raviart-Thomas space then $\nabla \boldsymbol{v}_1 \boldsymbol{n}_1$ and $\nabla \boldsymbol{v}_2 \boldsymbol{n}_2$ are constants. Therefore, we have the result.

Proof of lemma 2.3.6:

I now recall Green's identity. Let τ be an element in the partition $\mathcal{T}_h$.

$$(v, \nabla \cdot \boldsymbol{w})_\tau = -(\nabla v, \boldsymbol{w})_\tau + (v, \boldsymbol{w} \cdot \boldsymbol{n})_{\partial\tau} \tag{A.7}$$

The proof of (2.22) is obtained by summing up Green's identity on all the elements in the mesh. The boundary terms are then:

$$\sum_{\tau \in \mathcal{T}_h} \int_{\partial \mathcal{T}_h} v|_\tau \boldsymbol{w}|_\tau \cdot \boldsymbol{n}_\tau =$$

$$\sum_{F \in \mathcal{F}_i} \int_F (v|_{\tau_1} \boldsymbol{w}|_{\tau_1} \cdot \boldsymbol{n}_F - v|_{\tau_2} \boldsymbol{w}|_{\tau_2} \cdot \boldsymbol{n}_F) + \sum_{F \in \mathcal{F}_e} \int_F v|_\tau \boldsymbol{w}|_\tau \cdot \boldsymbol{n}_F =$$

$$\sum_{F \in \mathcal{F}_i} \int_F ((v|_{\tau_1} \boldsymbol{n}_F - v|_{\tau_2} \boldsymbol{n}_F) \cdot \frac{\boldsymbol{w}|_{\tau_1} + \boldsymbol{w}|_{\tau_2}}{2} + \frac{v|_{\tau_1} + v|_{\tau_2}}{2} (\boldsymbol{w}|_{\tau_1} - \boldsymbol{w}|_{\tau_2}) \cdot \boldsymbol{n}_F) +$$

$$\sum_{F \in \mathcal{F}_e} \int_F (v|_\tau \boldsymbol{n}_F \cdot \boldsymbol{w}|_\tau) =$$

$$\sum_{F \in \mathcal{F}_i} ([v], \{\boldsymbol{w}\})_F + \sum_{F \in \mathcal{F}_h} (\{v\}, [\boldsymbol{w}])_F$$

For (2.23) by definition

$$(\nabla \boldsymbol{v}, \nabla \boldsymbol{w})_{\mathcal{T}_h} = \sum_{i,j=1}^{d} (\partial x_j v_i, \partial x_j w_i)_{\mathcal{T}_h}.$$

using integration by parts for real functions

$$= \sum_{i,j=1}^{d} -(w_i, \partial^2 x_j v_i)_{\mathcal{T}_h} + \sum_{i,j=1}^{d} (\partial x_j v_i, w_i)_{\partial \mathcal{T}_h}$$

$$= \sum_{i,j} \sum_{\tau \in \mathcal{T}_h} \int_{\partial \mathcal{T}_h} \partial x_j v_i w_i n|_\tau$$

$$= \sum_{i,j} (\sum_{F \in \mathscr{F}_i} ([\partial x_j v_i], \{w_i\})_F + \sum_{F \in \mathscr{F}_h} (\{\partial x_j v_i\}, [w_i]))_F$$

On the other hand,

$$(\{\nabla \boldsymbol{v}\}, [[\boldsymbol{w}]])_{\mathscr{F}_h} = \sum_{F \in \mathscr{F}_h} (\{\nabla \boldsymbol{v}\}, [[\boldsymbol{w}]])_F$$

$$= \left(\begin{bmatrix} \{\partial x_1 v_1\} & \partial\{x_2 v_1\} \\ \{\partial x_1 v_2\} & \{\partial x_2 v_2\} \end{bmatrix}, \begin{bmatrix} [w_1] n_{11} & [w_1] n_{12} \\ [w_2] n_{11} & [w_2] n_{12} \end{bmatrix} \right)$$

$$= \sum_{i,j} \sum_{F \in \mathscr{F}_h} (\{\partial x_j v_i\}, [w_i]))_F.$$

Also

$$([[\nabla \boldsymbol{v}]], \{\boldsymbol{w}\})_{\mathscr{F}_i} = \left(\begin{bmatrix} [\partial x_1 v_1] n_{11} + [\partial x_2 v_1] n_{12} \\ [\partial x_1 v_2] n_{11} + [\partial x_2 v_2] n_{12} \end{bmatrix}, \{\boldsymbol{w}\} \right)_{\mathscr{F}_i}$$

$$= \sum_{i,j} \sum_{F \in \mathscr{F}_i} ([\partial x_j v_i], \{w_i\})_F$$

Therefore the third equality is proved. the fourth one is obtained from

$$[[\boldsymbol{v}]] : \{\nabla \boldsymbol{w}\} = [[\boldsymbol{v}]]\boldsymbol{n}_F \cdot \{\nabla \boldsymbol{w}\}\boldsymbol{n}_F.$$

Proof of lemma 2.3.9:

The proof is given in 2D. The proof in 3D is similar.

First, we consider the interior face F:

$$||[[\bar{\boldsymbol{v}_h}]]\boldsymbol{n}_F||_F = \frac{1}{|F|}|| \begin{bmatrix} n_{11}\int(v_{11}-v_{21}) & n_{12}\int(v_{11}-v_{21}) \\ n_{11}\int(v_{12}-v_{22}) & n_{12}\int(v_{12}-v_{22}) \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} ||_F$$

$$= \frac{1}{F}(||n_{11}^2 \int (v_{11}-v_{21}) + n_{12}^2 \int (v_{11}-v_{12})||_F^2 + ||n_{11}^2 \int (v_{12}-v_{22}) + n_{12}^2 \int (v_{12}-v_{22})||_F^2$$

$$= || \int (v_{11}-v_{21})||_F^2 + || \int (v_{12}-v_{22})||_F^2$$

which is equal to

$$||n_{11} \int (v_{11}-v_{21})||_F^2 + ||n_{12} \int (v_{11}-v_{21})||_F^2 + ||n_{11} \int (v_{12}-v_{22})||_F^2 + ||n_{12} \int (v_{12}-v_{22})||_F^2$$

$$= ||[[\bar{\boldsymbol{v}}]]||_F^2$$

Next, we consider the boundary face F:

$$||[[\bar{\boldsymbol{v}}]]\boldsymbol{n}||_F^2 = \frac{1}{|F|}|| \begin{bmatrix} n_{11}\int v_{11} & n_{12}\int v_{11} \\ n_{11}\int v_{21} & n_{12}\int v_{21} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \end{bmatrix} ||_F^2$$

$$= \frac{1}{|F|}|| \begin{bmatrix} n_{11}^2\int v_{11} + n_{12}^2\int v_{11} \\ n_{11}^2\int v_{21} + n_{12}^2\int v_{21}||_F^2 \end{bmatrix}$$

$$= || \int v_{11}||_F^2 + || \int v_{21}||_F^2$$

$$= ||n_{11} \int v_{11}||_F^2 + ||n_{12} \int v_{11}||_F^2 + ||n_{11} \int v_{21}||_F^2 + ||n_{12} \int v_{21}||_F^2$$

which is $||[[\bar{\boldsymbol{v}}]]||_F^2$.

Proof of lemma 2.3.10:

The proof is given in [5]. For completeness we recall it here.

Let $\boldsymbol{v}_h \in H^1(\mathcal{T}_h)$. According to vectorial version of (Lemma 2.1 in [5]: formula 2.7) if for fix face F $\pi_F^0 \boldsymbol{v}_h = \frac{1}{|F|}\int_F \boldsymbol{v}_h ds$ then,

$$
\begin{aligned}
&\frac{1}{|F|} \parallel [[v_h]] \parallel_{L_2(F)}^2 \\
&\leq C(\frac{1}{|F|} \parallel \pi_F^0[[\boldsymbol{v}_h]] \parallel_F^2 + \frac{1}{|F|} \parallel [[\boldsymbol{v}_h]] - \pi_F^0[[v_h]] \parallel_F^2) \\
\leq \quad & C\frac{1}{|F|} \parallel \pi_F^0[[\boldsymbol{v}_h]] \parallel_F^2 + C\sum_{\tau\in\mathcal{T}_h} \frac{1}{|F|} \parallel \boldsymbol{v}_h|_\tau - \pi_F^0 v_h|_\tau \parallel_F^2 \\
&\leq C'(\frac{1}{|F|} \parallel \pi_F^0[[\boldsymbol{v}_h]] \parallel_F^2 + \sum_{\tau\in\mathcal{T}_h} \parallel \nabla \boldsymbol{v}_h \parallel_{\mathcal{T}_h}^2).
\end{aligned}
$$

for some constant C' independent of h and using trace inequality in 2.29.

Using the fact that $|F| = \tilde{h}$ and $\tilde{h}$ is a function and summing over all faces we obtain

$$
\tilde{h}^{-1} \sum_{F\in\mathscr{F}_h} ||[[\boldsymbol{v}_h]]||_{L_2(F)}^2 \leq C'(\tilde{h}^{-1} \sum_{F\in\mathscr{F}_h} ||[[\bar{\boldsymbol{v}_h}]]||_{L_2(F)}^2 + N_{\mathscr{F}_h}||\nabla \boldsymbol{v}_h||_{\mathcal{T}_h}^2) \tag{A.8}
$$

$$
\leq C(||\tilde{h}^{-\frac{1}{2}}[[\boldsymbol{v}_h]]||_{\mathscr{F}_h}^2 + ||\nabla \boldsymbol{v}_h||_{\mathcal{T}_h}^2) \quad \text{by choosing} \quad c = c' N_{\mathscr{F}_h} \tag{A.9}
$$

For the vectorial case, we consider $d = 2$. The case $d = 3$ is similar.

when $\boldsymbol{v} \in \boldsymbol{H}^1(\mathcal{T}_h)$ if $\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then,

$$[[\boldsymbol{v}]] = \begin{bmatrix} [v_1]^T \\ [v_2]^T \end{bmatrix} \quad \text{therefore}$$

$$||[\boldsymbol{v}]||^2 = ||[v_1]^T||^2 + ||[v_2]^T||^2 = ||[v_1]||^2 + ||v_2||^2 \quad \text{and}$$

$$\begin{aligned} ||\nabla \boldsymbol{v}||^2 &= || \begin{bmatrix} \nabla v_1^T \\ \nabla v_2^T \end{bmatrix} || \\ &= ||\nabla v_1^T||^2 + ||\nabla v_2^T||^2 \\ &= ||\nabla v_1||^2 + ||\nabla v_2||^2 \end{aligned}$$

Therefore, by using (A.8) for v_1, v_2 and add them up we have proved (2.39)

Proof of the 2.3.11:

The proof of (2.40) is given in [26]. The proof of (2.41) follows from (2.39) and (2.40).

Proof of the 2.3.12:

Proof of the left inequality in (2.42) follows from (2.39). The right hand side inequality in (2.42) is obtained by (2.31).

Proof of 2.3.13:

The proof is given in [9]. For completeness we recall it here.

I first prove the priori estimate. Firstly, by the trace inequality

$$||[[\boldsymbol{\phi}_h]]||^2_{\mathscr{F}_h} \leq C_T^2 \tilde{h}^{-1} ||\boldsymbol{\phi}_h||^2_{\mathcal{T}_h}$$

or

$$||\tilde{h}^{-\frac{1}{2}}[[\boldsymbol{\phi}_h]]||^2_{\mathscr{F}_h} \leq C||\tilde{h}^{-1}\boldsymbol{\phi}_h||^2_{\mathcal{T}_h}.$$

Since $\boldsymbol{\pi}_0$ is a projection and using (2.34)

$$\begin{aligned} ||\tilde{h}^{-1}\boldsymbol{\phi}_h||^2_{\mathcal{T}_h} &= ||\tilde{h}^{-1}\boldsymbol{\pi}_0\boldsymbol{\phi}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{-1}(\boldsymbol{\phi}_h - \boldsymbol{\pi}_0\boldsymbol{\phi}_h)||^2_{\mathcal{T}_h} \\ &\leq ||\tilde{h}^{-1}\boldsymbol{a}_h||^2_{\mathcal{T}_h} + C_*||\nabla\boldsymbol{\phi}_h||^2_{\mathcal{T}_h} \end{aligned}$$

for some constant $C_* > 0$. By integration by parts,

$$\begin{aligned} \| \nabla\boldsymbol{\phi}_h \|^2_{\mathcal{T}_h} &= (\nabla\boldsymbol{\phi}_h, \nabla\boldsymbol{\phi}_h)_{\mathcal{T}_h} \\ &= -(\Delta\boldsymbol{\phi}_h, \boldsymbol{\phi}_h)_{\mathcal{T}_h} + (\{\nabla\boldsymbol{\phi}_h\}\boldsymbol{n}_F, [[\boldsymbol{\phi}_h]]\boldsymbol{n}_F)_{\mathscr{F}_h} + ([\nabla\boldsymbol{\phi}_h], \{\boldsymbol{\phi}_h\})_{\mathcal{F}_i} \end{aligned}$$

Since $\{\nabla\boldsymbol{\phi}_h\}\boldsymbol{n}_h$, $[\nabla\boldsymbol{\phi}_h]$ are constant (2.3.4) and $\Delta\boldsymbol{\phi}_h \in \boldsymbol{V}^0_h$, by definition of $\boldsymbol{\pi}_0$ and average we obtain,

$$\| \nabla\boldsymbol{\phi}_h \|^2_{\mathcal{T}_h} = -(\Delta\boldsymbol{\phi}_h, \boldsymbol{\pi}_0\boldsymbol{\phi}_h)_{\mathcal{T}_h} + (\{\nabla\boldsymbol{\phi}_h\}\boldsymbol{n}_F, [[\bar{\boldsymbol{\phi}_h}]]\boldsymbol{n}_F)_{\mathscr{F}_h} + ([\nabla\boldsymbol{\phi}_h], \{\bar{\boldsymbol{\phi}_h}\})_{\mathcal{F}_i}.$$

Since $\boldsymbol{b}_h \in \boldsymbol{W}^0_h$, $([[\bar{\boldsymbol{\phi}_h}]]\boldsymbol{n}_F, \boldsymbol{b}_h)_{\mathscr{F}_h} = ([[\boldsymbol{\phi}_h]]\boldsymbol{n}_F, \boldsymbol{b}_h)_{\mathscr{F}_h}$ therefore using Cauchy-Schwarz, the inverse, the trace and young's inequalities for each term we obtain

$$\begin{aligned} -(\Delta\boldsymbol{\phi}_h, \boldsymbol{a}_h)_{\mathcal{T}_h} &\leq C_I||\nabla\boldsymbol{\phi}_h||_{\mathcal{T}_h}||\tilde{h}^{-1}\boldsymbol{a}_h||_{\mathcal{T}_h} \\ &\leq \frac{1}{4}||\nabla\boldsymbol{\phi}_h||^2_{\mathcal{T}_h} + C_I^2||\tilde{h}^{-1}\boldsymbol{a}_h||^2_{\mathcal{T}_h} \\ ([[\bar{\boldsymbol{\phi}_h}]], \boldsymbol{b}_h)_{\mathscr{F}_h} &\leq C_T||\tilde{h}^{-1}\boldsymbol{\phi}_h||_{\mathcal{T}_h}||\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h||_{\mathscr{F}_h} \\ &\leq \frac{1}{4}||\nabla\boldsymbol{\phi}_h||^2_{\mathcal{T}_h} + \frac{1}{4C_*}||\tilde{h}^{-1}\boldsymbol{a}_h||^2_{\mathcal{T}_h} + C_*C_T^2||\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h||^2_{\mathscr{F}_h} \\ ([[\nabla\boldsymbol{\phi}_h]], \boldsymbol{c}_h)_{\mathcal{F}_i} &\leq C_T||\nabla\boldsymbol{\phi}_h||_{\mathcal{T}_h}||\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h||_{\mathcal{F}_i} \\ &\leq \frac{1}{4}||\nabla\boldsymbol{\phi}_h||^2_{\mathcal{T}_h} + C_T^2||\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h||^2_{\mathcal{F}_i} \end{aligned}$$

and thus

$$||\nabla \boldsymbol{\phi}_h||^2_{\mathcal{T}_h} \leq M(||\tilde{h}^{-1}\boldsymbol{a}_h||^2_{\mathcal{T}_h} + ||\tilde{h}^{\frac{1}{2}}\boldsymbol{b}_h||^2_{\mathscr{F}} + ||\tilde{h}^{-\frac{1}{2}}\boldsymbol{c}_h||^2_{\mathcal{F}_i})$$

where, $M = \max\{4C_I^2 + \frac{1}{C_*}, 4C_*C_T^2, 4C_T^2\}$

$$\| \, [[\boldsymbol{\phi}_h \boldsymbol{n}_F]] \, \| = \| \, [[\boldsymbol{\phi}_h]] \, \| \, .$$

Since (2.43) is a square linear system of size $d(N_{\mathcal{T}_h} + N_{\mathscr{F}} + N_{\mathcal{F}_i})$, existence and uniqueness is equivalent. Let us denote the system by $Aw = b$ and suppose w_1, w_2 are the solutions of the system and let $e = w_1 - w_2$. then, $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} = 0$ and from priori estimate $e = 0$.

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